

CSE 21—Mathematics for Algorithm and System Analysis

Spring, 2006

May 2, 2006

Day 9

Exponential Generating Functions

Instructor: Neil Rhodes

Example

How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

- Generating function
- Simplify
- Find coefficient

Exponential Generating Functions

Used to model problems involving arrangements and distributions of distinct objects

Example problem

- find number of different arrangements of four letters, choosing from a's, b's, and c's with at least two a's.

Solution

- Four letter combinations:
 - {a, a, a, a}: $4!/4!0!0!$
 - {a, a, a, b}: $4!/3!1!0!$
 - {a, a, a, c}: $4!/3!0!1!$
 - {a, a, b, b}: $4!/2!2!0!$
 - {a, a, b, c}: $4!/2!1!1!$
 - {a, a, c, c}: $4!/2!0!2!$
- Total is sum, so we want that sum as the coefficient of x^4
- Ordinary generating function can solve number of integer solutions to:
 - $e_1 + e_2 + e_3 = 4 \quad e_1 \geq 2 \quad e_2, e_3 \geq 0$
- But, we want each solution to above problem to contribute:
 - $(e_1+e_2+e_3)!/e_1!e_2!e_3!$

Exponential Generating Function

Function of form:

$$g(x) = a_0 + a_1x + \frac{a_2x^2}{2!} + \frac{a_3x^3}{3!} + \cdots + \frac{a_rx^r}{r!} + \cdots$$

Example

$$g(x) = \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^2$$

- Coefficient of x_r in $g(x)$ is equal to: $\sum_{e_1+e_2+e_3=r} \frac{1}{e_1!e_2!e_3!}$
- Coefficient of $x_r/r!$ in $g(x)$ is equal to: $\sum_{e_1+e_2+e_3=r} \frac{r!}{e_1!e_2!e_3!}$

Example

Find the exponential generating function for a_r , the number of different arrangements of r objects chosen from four different type of objects with each type of object appearing at least 2 and no more than 5 times

Example

Find the number of ways to place r (distinct) people into three rooms with at least one person in each room

Finding Desired Coefficients

Useful identities

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

$$e^{nx} = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^3x^3}{3!} + \cdots + \frac{n^rx^r}{r!} + \cdots$$

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots +$$

$$\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots +$$

Example

Find the number of different r arrangements of objects chosen from unlimited supplies of n types of objects

- Generating function
- Simplify
- Find coefficient

Example

Find the number of ways to place 25 people into 3 rooms with at least one person in each room

- Generating function
- Simplify
- Find coefficient

Example

Find the number of r -digit quaternary sequences (digits: 0, 1, 2, 3) with an even number of 0's and an odd number of 1's

- Generating function
- Simplify
- Find coefficient