

CSE 21—Mathematics for Algorithm and System Analysis

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Day 8

Generating Functions

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Outline

What a generating function is

How to create a generating function to model a problem

Finding the desired coefficient

(Ordinary) Generating Functions

A different way to solve problems involving selection and arrangement problems with repetition

Can easily support special constraints not easy to do otherwise

- Find the number of ways to distribute 70 identical objects into 9 boxes with an odd number between 3 and 7 in the first box, an even number between 4 and 20 in the second box, and at most 3 in the other boxes.

Idea

- Create a generating function $g(x)$ which models the problem of selecting r objects. The coefficient of x^r in $g(x)$ counts the number of ways to select r objects
- $g(x) = a_0 + a_1x + a_2x^2 + \dots a_rx^r + \dots + a_nx^n$

Example

- $g(x) = (1+x)^n$ models the number of ways to select r objects from n objects

$$g(x) = (1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n$$

Let's look at the coefficients of x^5 in $(1+x+x^2)^4$

- Ways to create x^5 :
 - $x^0x^2x^2x^1$
 - $x^1x^1x^1x^2$
 - etc.
- In general, $x^{e_1}x^{e_2}x^{e_3}x^{e_4}$
 - where $0 \leq e_1, e_2, e_3, e_4 \leq 2$
- Can be looked at as the number of integer solutions to:
 - $e_1 + e_2 + e_3 + e_4 = 5$ (where $0 \leq e_1, e_2, e_3, e_4 \leq 2$)
- Which can also be looked at as the number of ways to place 5 balls in four boxes with at most 2 balls in each box
- Which can also be looked at as the number of ways to select 5 objects from a collection of 4 different types of objects (with two identical objects of each type).

$g(x) = (1+x+x^2)^4$ is a generating function for the above problems

- Actually, $g(x)$ is a generating function for the problem with any number, r , of {balls, objects}, not just 5
- We always write a general generating function for any r

Example

Find a generating function for a_r , the number of ways to select r balls from a pile of three green, three white, three blue, and three gold balls

- Same as number of integer solutions to:
 - $e_1 + e_2 + e_3 + e_4 = r \quad 0 \leq e_i \leq 3$

Example

Find a generating function to model counting all selections of six objects chosen from three types of objects with:

- repetition of up to four objects of each type
- unlimited repetition

Example

Find a generating function to model:

- the number of ways to distribute r identical objects into five boxes with an even number of objects not exceeding 10 in the first two boxes and between 3 and 5 in the other boxes.

Example

Find a generating function for a_r , the number of ways a roll of six distinct dice can show a sum of r if

- The first three dice are odd and the second three even
- The i th die does not show a value of i

Finding the Desired Coefficient

Reduce complicated generating function to simple function or product of simple functions

- The simple functions:

$$\frac{1 - x^{m+1}}{1 - x} = 1 + x + x^2 + \cdots + x^m$$

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots$$

Finding the Desired Coefficient

Reduce complicated generating function to simple function or product of simple functions

- The simple functions:

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n$$

$$(1-x^m)^n = 1 - \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + (-1)^k \binom{n}{k}x^{km} + \cdots + \binom{n}{n}x^{nm}$$

$$\frac{1}{(1-x)^n} = 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \cdots + \binom{r+n-1}{r}x^r + \cdots$$

Product of Generating Functions

If

- $f(x) = a_0 + a_1x + a_2x^2 + \dots$
- $g(x) = b_0 + b_1x + b_2x^2 + \dots$

and

- $h(x) = f(x)g(x)$

then

- $$\begin{aligned} h(x) = & a_0b_0 + \\ & (a_0b_1 + a_1b_0)x + \\ & (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \\ & \dots + \\ & (a_0b_r + a_1b_{r-1} + \dots + a_{r-1}b_1 + a_rb_0)x^r + \\ & \dots \end{aligned}$$

Example

Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \dots)^5$

- Rewrite as product of simpler functions
- Final simplification:
- Find coefficient

Example

Find the number of ways to collect \$15 from 20 distinct people if each of the first 19 people can give \$0 or \$1 and the 20th person can give either \$0 or \$1 or \$5

- Generating function: $h(x) =$
- Simplify
- Find coefficient

Example

How many ways to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls?

- Generating function
- Simplify
- Find coefficient

Example

How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

- Generating function
- Simplify
- Find coefficient