

CSE 21—Mathematics for Algorithm and System Analysis

Spring, 2006

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Day 6

Inclusion/Exclusion

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Venn Diagrams

Set A in universe U

Venn Diagrams

Sets L and F in universe U

$$|L \cup F| = |L| + |F| - |L \cap F|$$

L = # students taking Latin

F = # students taking French

Example (from 8.1 example 2)

How many arrangements of the digits $0, 1, \dots, 9$ are there in which the first digit is greater than 1 and the last digit is less than 8?

Venn Diagram with Three Sets

Sets F, L, G in universe U

Number students not taking any language?

Inclusion-Exclusion Formula for 3 Sets

$$\begin{aligned} N(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) &= \\ N & \\ - [N(A_1) + N(A_2) + N(A_3)] & \\ + [(N(A_1 \cap A_2) + N(A_2 \cap A_3) + N(A_3 \cap A_1))] & \\ - N(A_1 \cap A_2 \cap A_3) & \\ = N - \sum_i N(A_i) + \sum_{ij} N(A_i \cap A_j) - N(A_1 \cap A_2 \cap A_3) & \end{aligned}$$

Example (from 8.1, example 4)

How many positive integers ≤ 70 are relatively prime to 70?

Example (from 8.1, exercise 31)

Suppose a school with 120 students offers Yoga and Karate. If the number of students taking Yoga alone is twice the number taking Karate (possibly, Karate and Yoga), if 25 more students study neither skill than study both skills, and if 75 students take at least one skill, then how many students study Yoga?

Inclusion-Exclusion Formula for n Sets

Let $A_1, A_2, \dots, A_n$ be sets in a universe U of N elements. Let S_k denote the sum of the sizes of all k -tuple intersections of the A_i s.

$$S_1 = \sum_i N(A_i)$$

$$S_2 = \sum_{ij} N(A_i \cap A_j)$$

S_k = sum of all intersections of exactly k A_i s.

$$S_n = N(A_1 \cap A_2 \dots \cap A_n)$$

Binomial Identity

Sum of odd terms = sum of even terms

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^k \binom{n}{k} + \cdots + (-1)^n \binom{n}{n}$$

- Proof by combinatorial argument

Inclusion-Exclusion Formula for n Sets

Let $A_1, A_2, \dots, A_n$ be sets in a universe U of N elements. Let S_k denote the sum of the sizes of all k -tuple intersections of the A_i s.

$$N(\bar{A}_1 \cap \bar{A}_2 \dots \cap \bar{A}_n) = N - S_1 + S_2 - S_3 + \dots + (-1)^k S_k + \dots + (-1)^n S_n$$

- number of times an element is counted which is in exactly m A_i s:

$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \dots + (-1)^k \binom{m}{k} + \dots + (-1)^m \binom{m}{m}$$

- number of times an element is counted which is in 0 A_i s:

Corollary:

$$N(A_1 \cup A_2 \dots \cup A_n) = S_1 - S_2 + S_3 - \dots + (-1)^{k-1} S_k + \dots + (-1)^{n-1} S_n$$

Example (from 8.2 example 1)

How many ways are there to select a 5-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

- A_1 = void in clubs
- A_2 = void in hearts
- A_3 = void in spades
- A_4 = void in diamonds
- Calculate $N-(A_1 \cup A_2 \cup A_3 \cup A_4)$
- $N - S_1 + S_2 - S_3 + S_4$

Example (from 8.2, example 3)

How many different integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20 \quad (0 \leq x_i \leq 8)$$

- A_i = set of integer solutions with $x_i > 8$

Example (from 8.2, example 2)

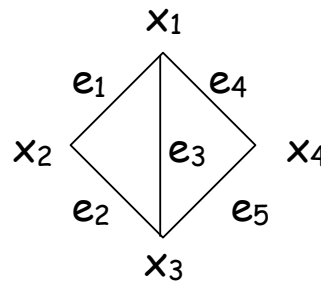
How many ways are there to distribute r distinct objects into five (distinct) boxes with at least one empty box?

- Let A_i = set of distributions with a void in box i .

Example (from 8.2, example 5)

How many ways are there to color the four vertices in a graph with n colors (such that the vertices with a common edge must be different colors)?

- Define A_i = all colorings in which either end of edge e_i have the same color.



- $N =$
- $S_1 =$
- $S_2 =$
- $S_3 =$
- $S_4 =$
- $S_5 =$
- Total =