

CSE 21—Mathematics for Algorithm and System Analysis

Spring, 2006

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Day 5

Binomial Coefficients

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Example 6 (from 5.4 example 6)

How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$ ($x_i \geq 0$)?

With $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

Example (similar to 5.4 example 8)

A bitonic sequence is a sequence of 1s followed by 0s followed by 1s, or a sequence of 0s followed by 1s followed by 0s. 010, 011110, 10 are bitonic sequences. How many bitonic sequences of length 9 are there?

Example (from 5.4 Exercise 20)

What fraction of all arrangements of EFFLORESCENCE has consecutive **C**s and consecutive **F**s but no consecutive **E**s?

All of These are Equivalent

1. The number of ways to select r objects with repetition from n different types of objects
2. The number of ways to distribute r identical objects into n distinct boxes
3. The number of nonnegative integer solutions to:
 - $x_1 + x_2 + \dots + x_n = r$.

Binomial Theorem

Let's look at $(x+y)^n$

- $(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form $x^{3-k}y^k$ in $(x+y)^3$?

How many ways to choose $3-k$ **x**s and k **y**s from 3 total?

- $C(3, k)$

Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots \binom{n}{n}y^n$$

Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof

Binomial Identities

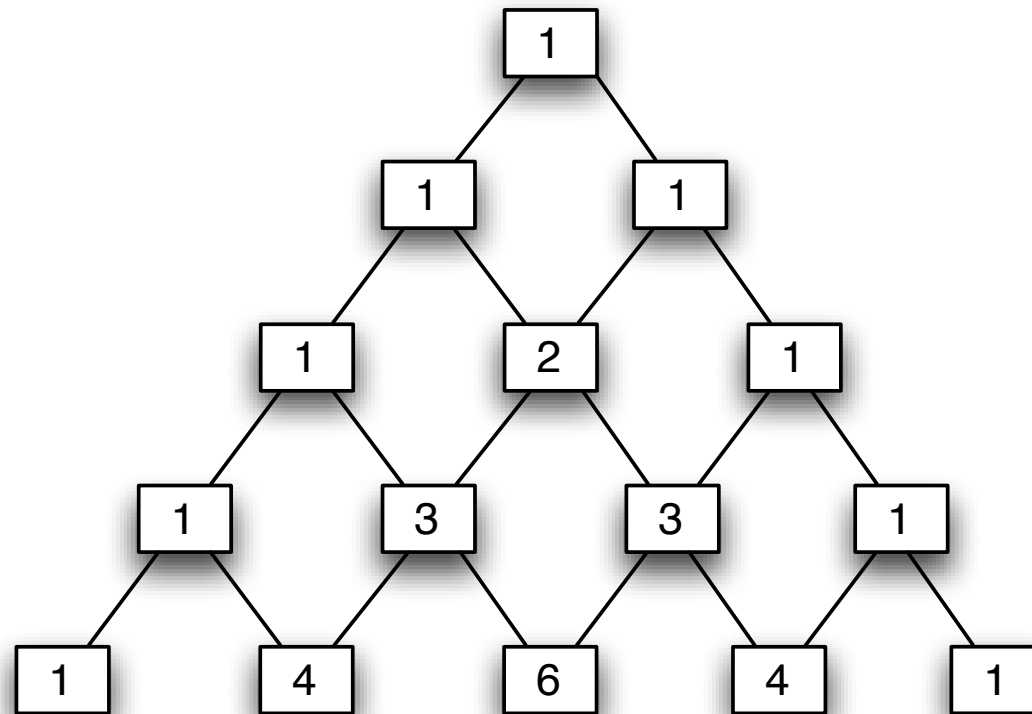
Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Proof by combinatorial argument

Pascal's Triangle

Number of paths equals $C(n, k)$

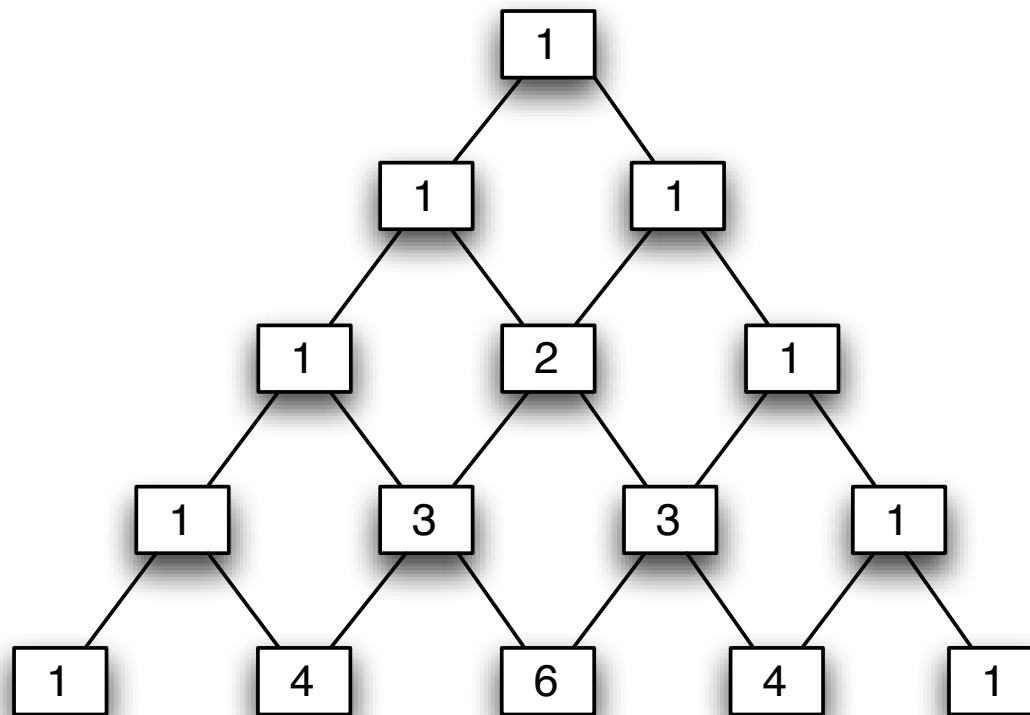


Binomial Identities

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Proof by Pascal's triangle (block-walking)



Other Binomial identities

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof by binomial theorem

Other Binomial identities

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof by induction:

Other Binomial identities

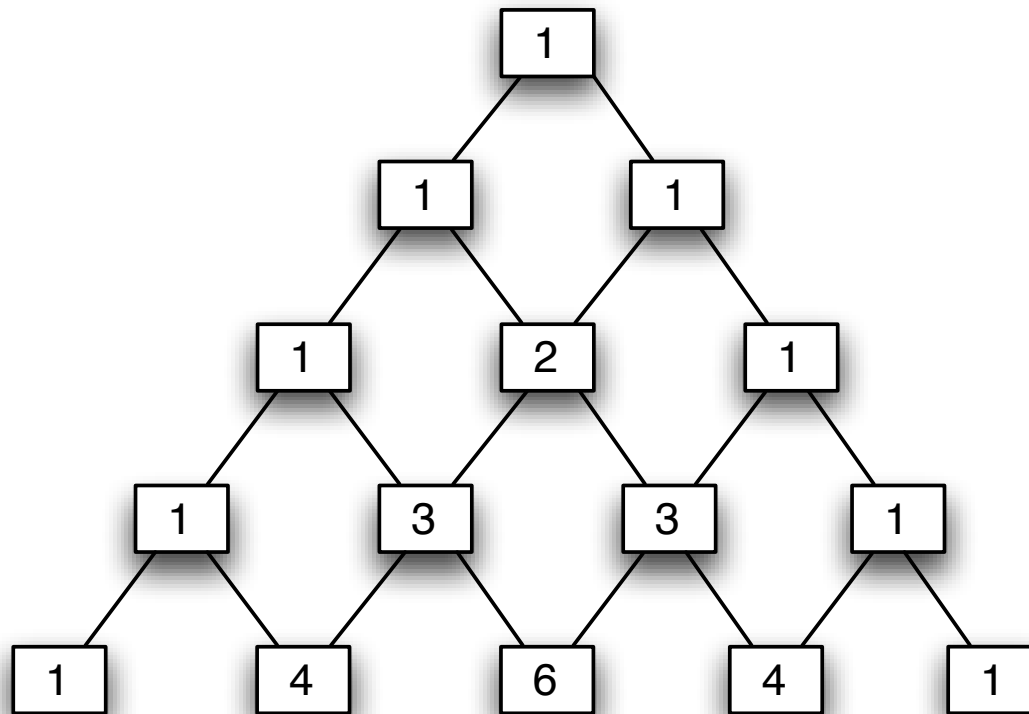
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof by combinatorial argument:

Other Binomial identities

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

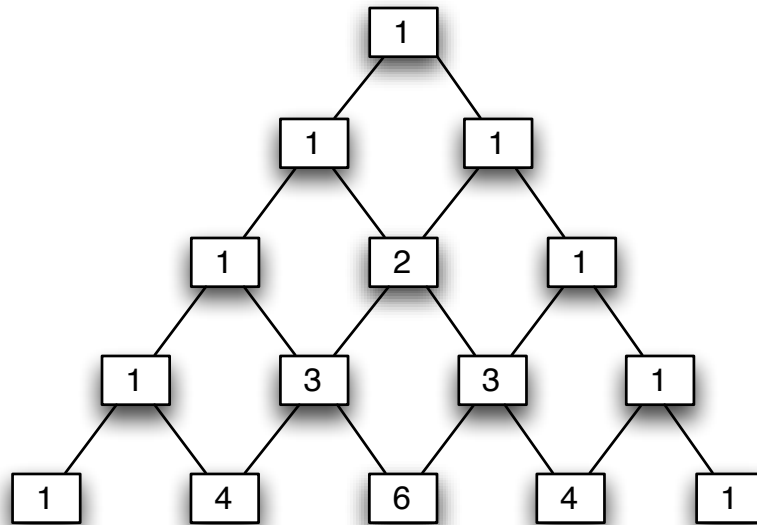
Proof by Pascal's triangle (block-walking):



Other Binomial identities

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

Proof by Pascal's triangle (block-walking)



Other Binomial identities

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

Proof by combinatorial argument

Other Binomial identities

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

Proof by combinatorial argument