

CSE 21—Mathematics for Algorithm and System Analysis

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Day 4

Distributions

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Example (from 5.3 Example 8)

How many arrangements of **b, a, n, a, n, a** such that:

- The **b** is immediately followed by an **a**?
- The pattern **bnn** never occurs.
- The **b** occurs before any of the **as**

Example (from 5.3 exercise 22)

How many ways are there first to pick a subset of r people from 50 people (each of a different height) and next to pick a second subset of s people such that everyone in the first subset is shorter than everyone in the second subset?

Distributions

Distributions of distinct objects

- Same as arrangements
- Distributing r distinct objects into n different boxes:
 - Put the r objects into a row
 - Stamp out n box names on each object

Distributions of identical objects

- Same as selections
- Distributing r identical objects into n different boxes
 - Choose a subset of r box names with repetition from n boxes
 - $C(r+n-1, r)$ distributions

Example (from 5.4 example 1)

How many ways are there to assign 100 different diplomats to five different countries?

If 20 diplomats must be assigned to each country?

Example

You're playing a card game and your two opponents have 4 spades between them. Which is more likely: that they're split 3-1, or 2-2?

Example (from 5.4 Example 2)

In a game of bridge (N, S, E, W each dealt 13 cards). What is the probability that West has all 13 Spades?

What is the probability that each hand has one Ace?

Example (from 5.4 example 4)

How many ways to distribute four identical oranges and six distinct apples into five distinct boxes?

In what fraction of these distributions does each box get exactly two objects?

Example (from 5.4 example 5)

Show the number of ways to distribute r identical balls into n distinct boxes with at least one ball in each box is $C(r-1, n-1)$.

With at least r_1 balls in the first box, ..., r_n balls in the n th box, the number is $C(r-r_1-r_2-\dots-r_n+n-1, n-1)$

Example 6 (from 5.4 example 6)

How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$ ($x_i \geq 0$)?

With $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

Example (similar to 5.4 example 8)

A bitonic sequence is a sequence of 1s followed by 0s followed by 1s, or a sequence of 0s followed by 1s followed by 0s. 010, 011110, 10 are bitonic sequences. How many bitonic sequences of length 9 are there?

Example (from 5.4 Exercise 20)

What fraction of all arrangements of EFFLORESCENCE has consecutive **C**s and consecutive **F**s but no consecutive **E**s?

All of These are Equivalent

1. The number of ways to select r objects with repetition from n different types of objects
2. The number of ways to distribute r identical objects into n distinct boxes
3. The number of nonnegative integer solutions to:
 - $x_1 + x_2 + \dots + x_n = r$.

Summary

Arrange/Select/Distribute r objects from n items

	Arrangement (ordered outcome) or Distribution of distinct objects	Combination (unordered outcome) or Distribution of identical objects
No repetition		
Unlimited repetition		
Restricted repetition		

Binomial Theorem

Let's look at $(x+y)^n$

- $(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form $x^{3-k}y^k$ in $(x+y)^3$?

How many ways to choose $3-k$ **x**s and k **y**s from 3 total?

- $C(3, k)$

Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots \binom{n}{n}y^n$$

Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof
- Proof by Pascal's triangle
- Proof by combinatorial argument

Pascal's Triangle

Number of paths equals $C(n, k)$

