

CSE 21—Mathematics for Algorithm and System Analysis

Summer, 2005

Day 2

Basic Enumeration and Counting

Instructor: Neil Rhodes

Permutations and Combinations

Permutation of n objects: ordering of all n objects

r -permutation: ordering of r of n objects

- Order matters
- $P(n, 2) = n(n-1)$
 $P(n, 3) = n(n-1)(n-2)$
 $P(n, r) = n(n-1)(n-2)\dots(n-(r-1))$
 $P(n, n) = n(n-1)\dots(2)(1) = n!$
- $P(n, r) = n(n-1)(n-2)\dots(n-(r-1)) = n!/(n-r)! = n^{\underline{r}} = (n)_r$

r -combination of n objects: choosing r of n objects (unordered)

- $C(n, r) = \# \text{ of } r\text{-permutations} / \# \text{ of ways to order } r \text{ items}$
 $= P(n, r) / P(r, r)$
 $= n!/(n-r)! / r!$
 $= n!/(n-r)!r!$
- $C(n, r) = C(n, n-r)$

$$\binom{n}{r} = C(n, r) = \frac{n(n-1)\dots(n-(r-1))}{r!} = \frac{n!}{r!(n-r)!}$$

Example Problem (from 5.2 example 1)

How many ways are there to rank n candidates for the job of chief wizard?

If rankings are chosen at random, what is the probability that the fifth candidate is in second place?

Example Problem (from 5.2 example 4)

If a 5-card hand is chosen at random, what is the probability of a flush (all 5 cards of the same suit)?

- How many 5-card hands are there?
- How many 5-card hands are flushes?

Example Problem (from 5.2 example 2)

How many ways are there to arrange the seven letters in the word **SYSTEMS**?

In how many do the three **S** letters appear next to each other?

In how many does the **E** occur somewhere before the **M**?

In how many do the three S letters appear next to each other *and* the **E** occurs somewhere before the **M**?

Example Problem (from 5.2 exercise 40)

What is the probability that a five-card hand has at least one card of each suit?

What is the probability that a six-card hand has at least one card of each suit?

Binomial Theorem

Let's look at $(x+y)^n$

- $(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form $x^{3-k}y^k$ in $(x+y)^3$?

How many ways to choose $3-k$ **x**s and k **y**s from 3 total?

- $C(3, k)$

Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots \binom{n}{n}y^n$$

Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof
- Proof by Pascal's triangle
- Proof by combinatorial argument

Pascal's Triangle

Number of paths equals $C(n, k)$

