

CSE 21—Mathematics for Algorithm and System Analysis

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Lecture 17

Graph Theory

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Graph

Definition

- **Graph** G consists of a pair (V, E)
 - V is a finite set called the **vertices**
 - E is a set of **edges** that join distinct pairs of V
- We say two vertices are **adjacent** if there is an edge between them
- We say a vertex is **incident** to the edges connecting to it

Example

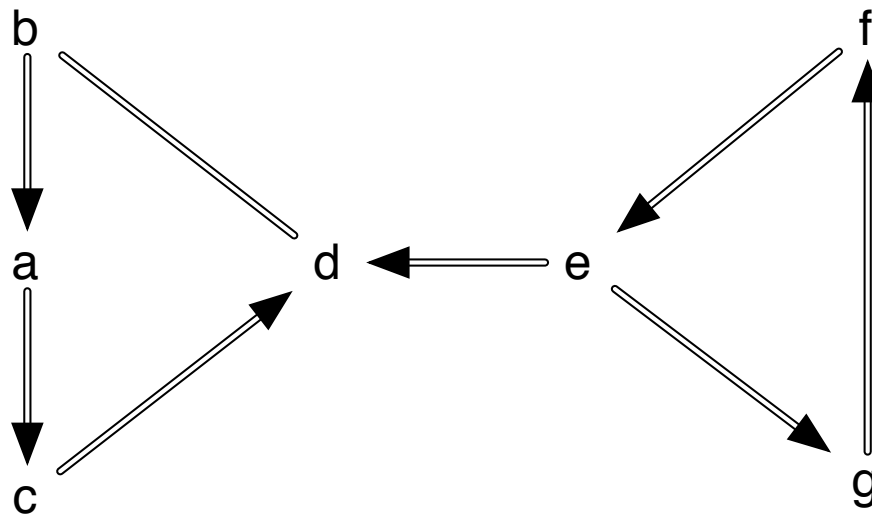
- $V = \{a, b, c, d\}$, $E = \{(a, b), (b, c), (c, d), (d, a), (a, c)\}$

Directed Graph (Digraph)

Definition

- Like a graph, but each edge is *directed*
- Can have one edge in each direction between a pair of vertices

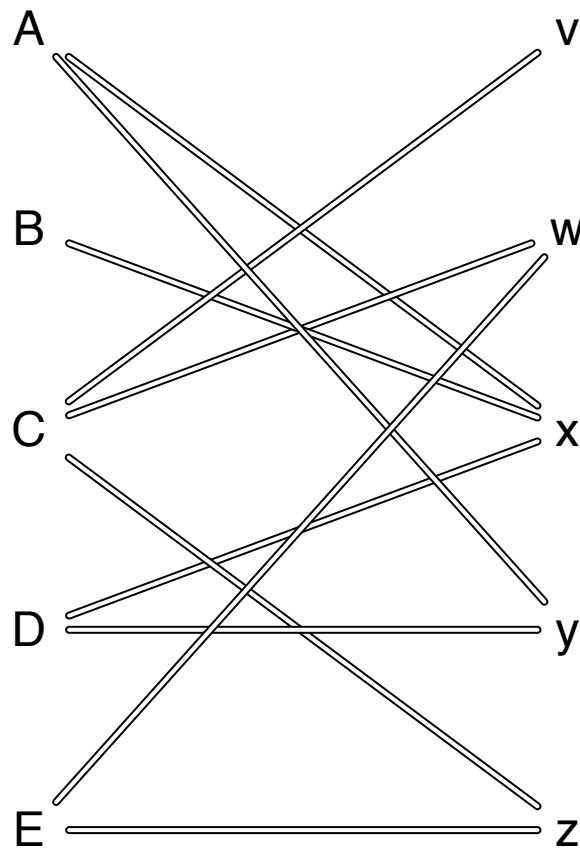
Example



Graph Example

A, B, C, D, and E are applying for jobs v, w, x, y, z as shown in graph G.

Can each person be assigned to a different job?



Definitions

Path

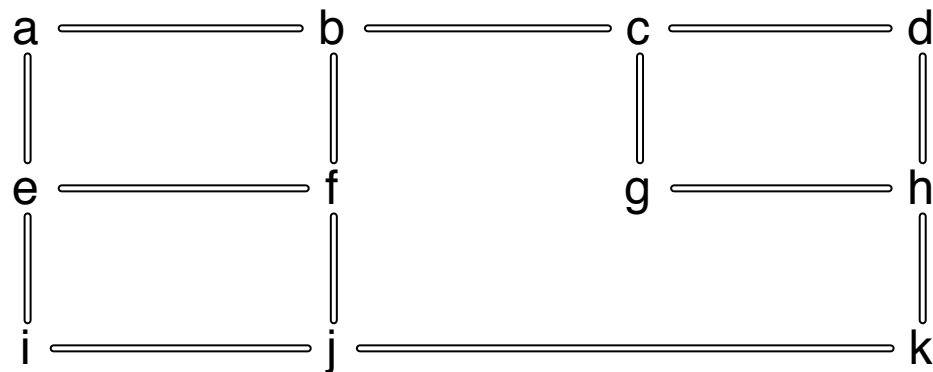
- A sequence of distinct vertices P_1 - P_2 -...- P_n where each pair of consecutive vertices has an edge joining them

Circuit

- A path from P_1 to P_n where there's also an edge from P_n to P_1 .

Connected

- A graph is connected if there's a path between all pairs of vertices

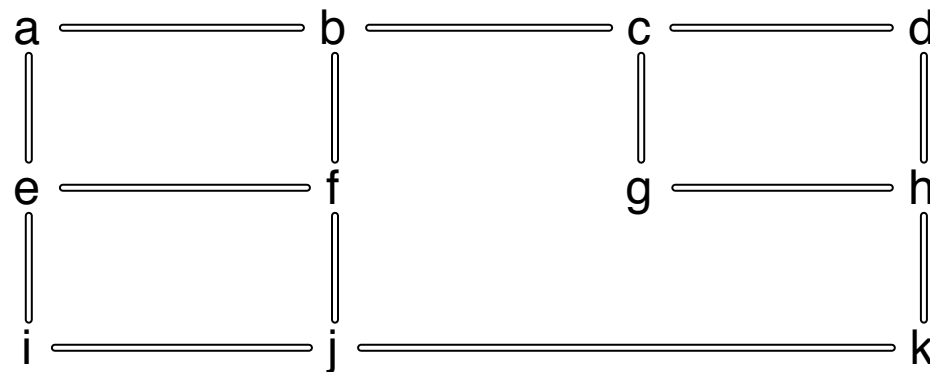


Network Reliability Example

Vertices are power stations

Edges are electrical distribution lines

- Which edges need to be removed in order to disconnect the (remaining) graph?
- Which vertices need to be removed in order to disconnect the (remaining) graph?



Definitions

Degree of a vertex

- The count of the number of edges adjacent to the vertex

Vertex cover of a graph G

- Subset of vertices G' such that every edge in G is adjacent to at least one vertex of G'

Edge Cover of a graph G

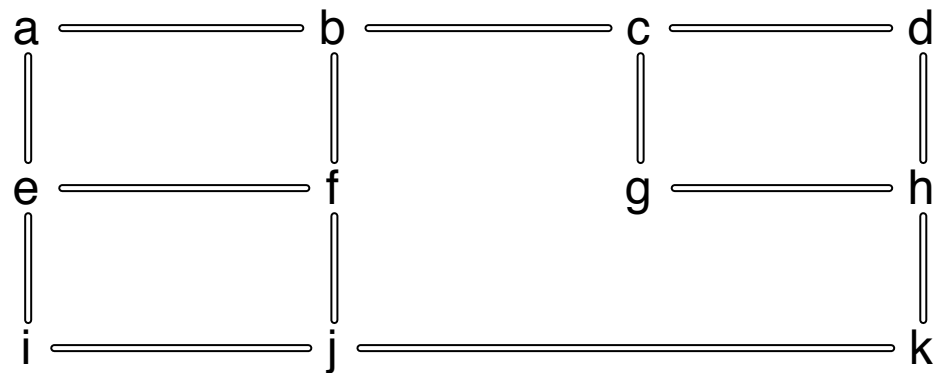
- Subset of edges E' in G such that every vertex in G is incident to at least one edge of E' .

Street Surveillance Example

Vertices are intersections

Edges are streets

- On what intersections should police officers be placed to patrol each edge under surveillance. (Each edge needs an adjacent police officer).



Scheduling Meetings Example

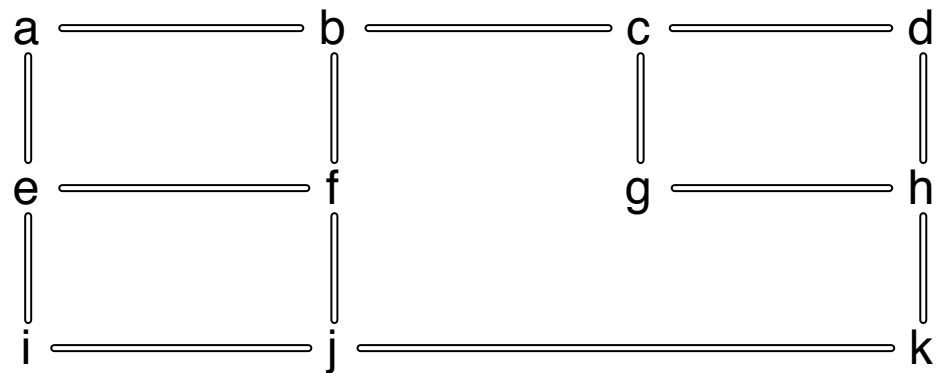
Vertices are committees

Edges are committees with overlapping membership

- What is the largest set of committees that can meet at the same time?

(Maximal) Independent set

- (Largest) subset of vertices in a graph with no edges between them.



Relationship between Independent Set and Vertex Cover

Given $G=(V, E)$

- I is Independent set iff $V-I$ is a vertex cover

Proof

- $\rightarrow$

- $\leftarrow$

Result

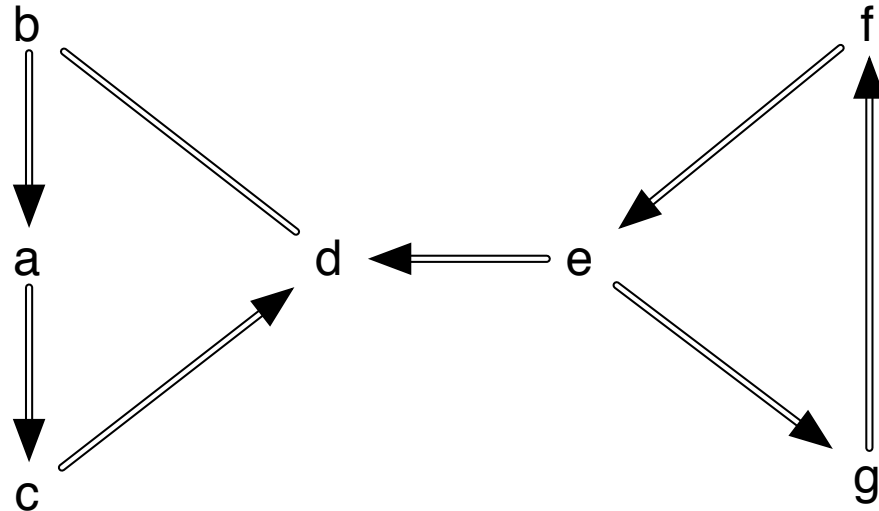
- Finding maximal independent set is same problem as finding minimal vertex cover

Influence Example

Vertices are people

Edge from i to j if i influences j

- What is a minimal set of influencers who can spread an idea to everyone?



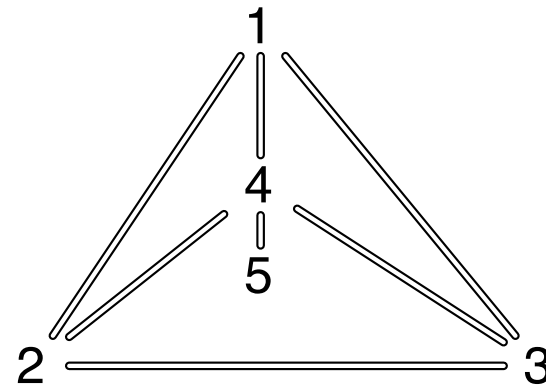
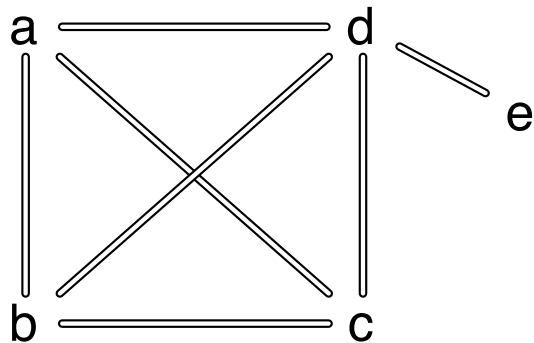
Isomorphism

G_1 is *isomorphic* to G_2 if:

- there is a one-to-one correspondence between V_1 and V_2 where two vertices are adjacent in G_1 iff the corresponding pair of vertices are adjacent in G_2
 - $\exists$ bijection $f: V_1 \rightarrow V_2$ such that $(x, y) \in E_1$ iff $(f(x), f(y)) \in E_2$

Intuitively:

- Can redraw G_1 so that it looks like G_2



Definitions

Subgraph G' of G

- Subset of edges and vertices of G

Complete graph, K_n of n vertices

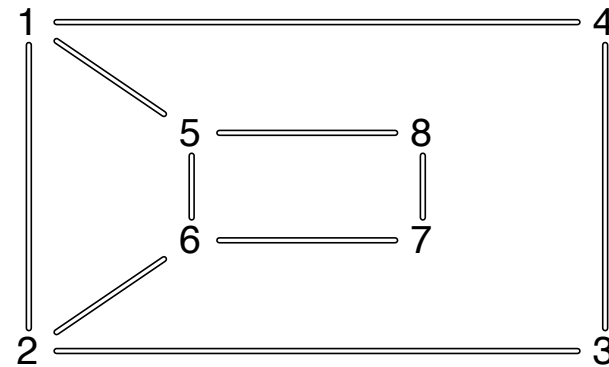
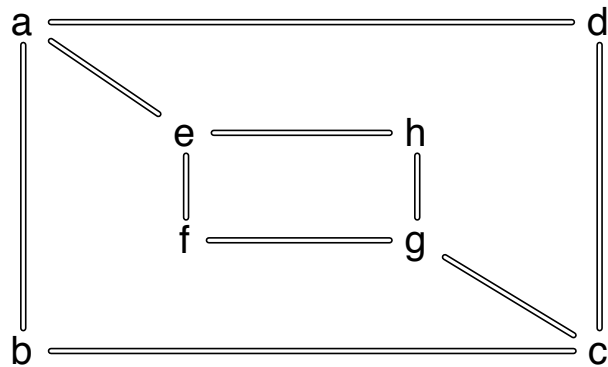
- Graph on n vertices with all vertices adjacent to each other

Isomorphism

Must match:

- Number of vertices
- Number of edges
- Degrees of vertices
- Subgraphs

Example:



Theorem

In any graph, the sum of the degrees of the vertices is equal to twice the number of edges

Corollary:

- The number of vertices of odd degree is even

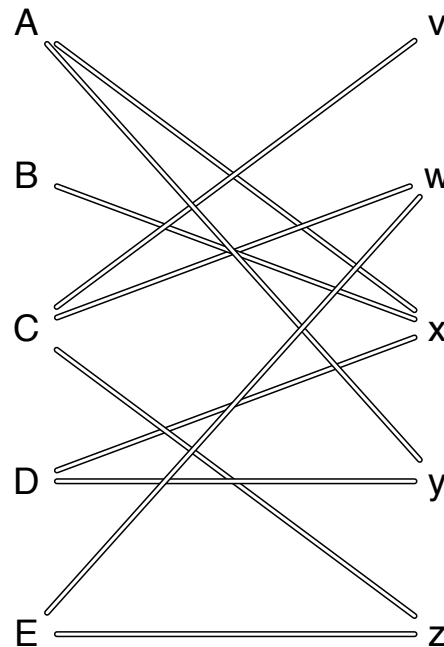
Bipartite graph

A graph whose vertices can be partitioned into two sets: V_1 and V_2 where:

- The only edges are between vertices in V_1 and V_2

Theorem:

- G is bipartite iff every circuit in G has even length

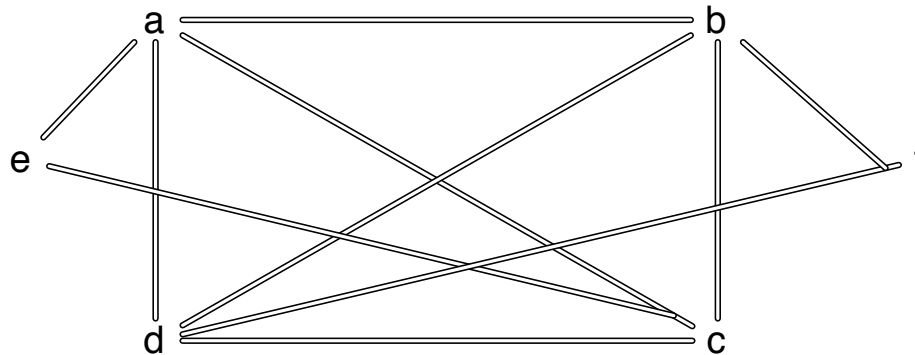


Planar graphs

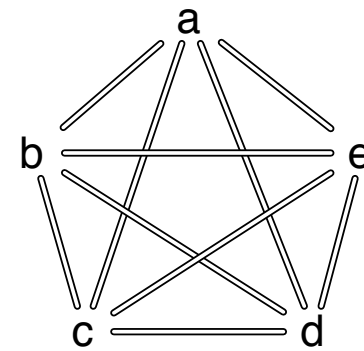
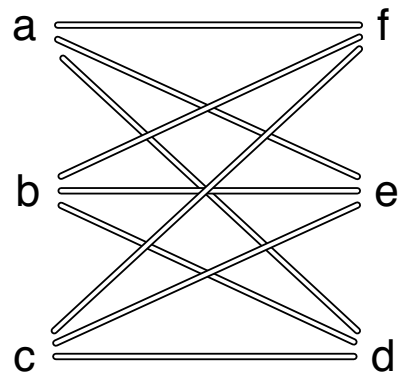
Graphs that can be drawn in the plane with no crossing edges

Plane graph

- A planar depiction of a planar graph



Some non-planar graphs



Theorem

A graph is planar iff it does not contain a subgraph that is a K_5 or $K_{3,3}$ configuration

Definition

- A $K_{3,3}$ (K_5) configuration is a graph that can be obtained from a $K_{3,3}$ (K_5) by adding vertices in the middle of some edges

Definitions

Multigraph

- We allow multiple edges between pairs of vertices (and allow loop edges between a single vertex)

Trail

- Like a path, sequence of vertices: $V_1-V_2-\dots-V_n$, except:
 - Vertices can be repeated
 - Edges cannot can be repeated

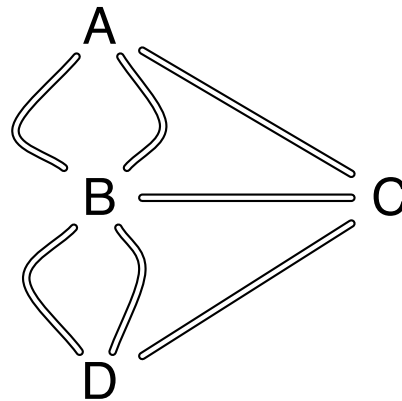
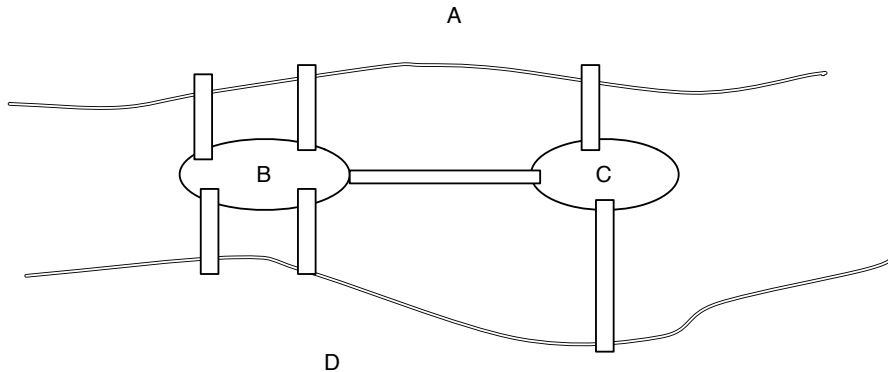
Cycle

- Like a trail, but with an edge (V_n, V_1)

Konigsberg bridges

Euler cycle: A cycle that

- Uses all vertices
- Contains all edges in the multigraph



First theorem of graph theory

A multigraph contains an Euler cycle iff:

- The graph is connected
- Each vertex has even degree

Corollary: A multigraph has an Euler trail iff:

- it is connected
- Exactly two vertices have odd degree