

CSE 21—Mathematics for Algorithm and System Analysis

Spring, 2006

May 18, 2006

Day 14

Linear Homogeneous Recurrence
Relations with Constant Coefficient

Instructor: Neil Rhodes

Examples of Limited Master Method

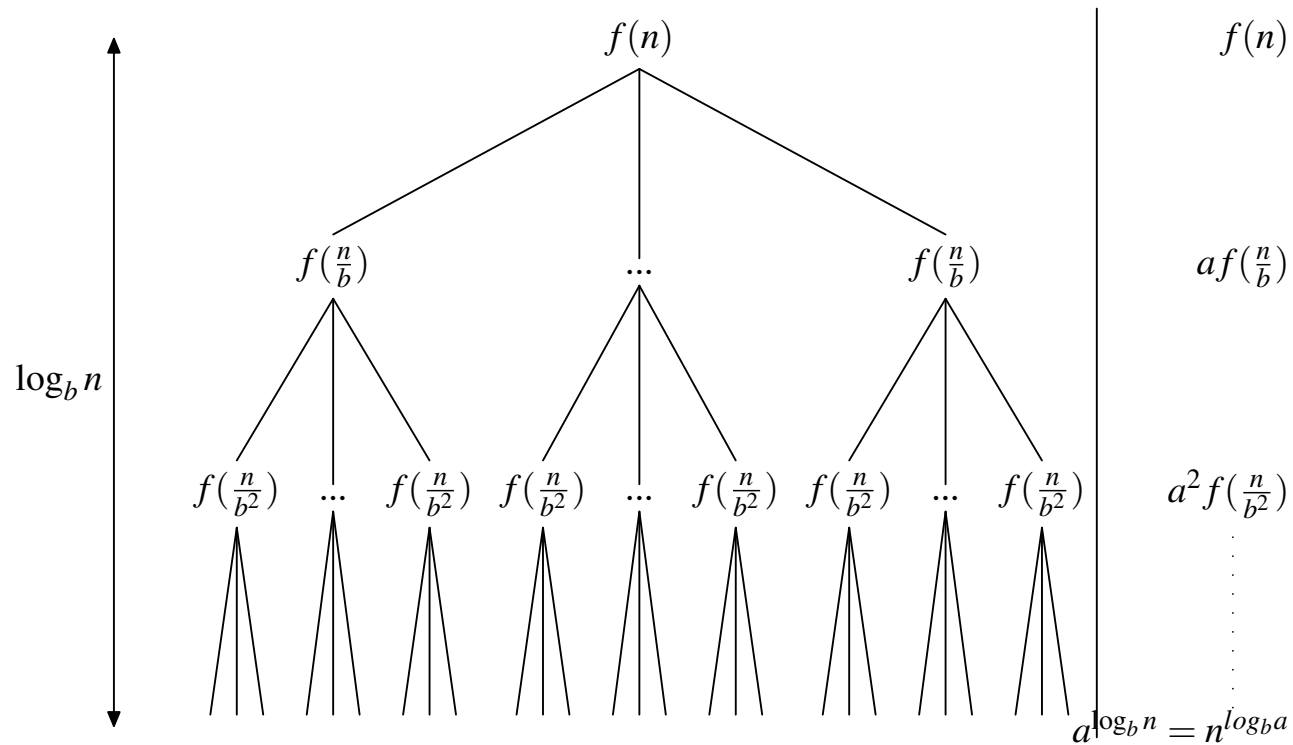
$$T_n = 3T_{n/2} + n$$

$$T_n = 4T_{n/4} + n$$

$$T_n = 4T_{n/5} + n$$

Master Method

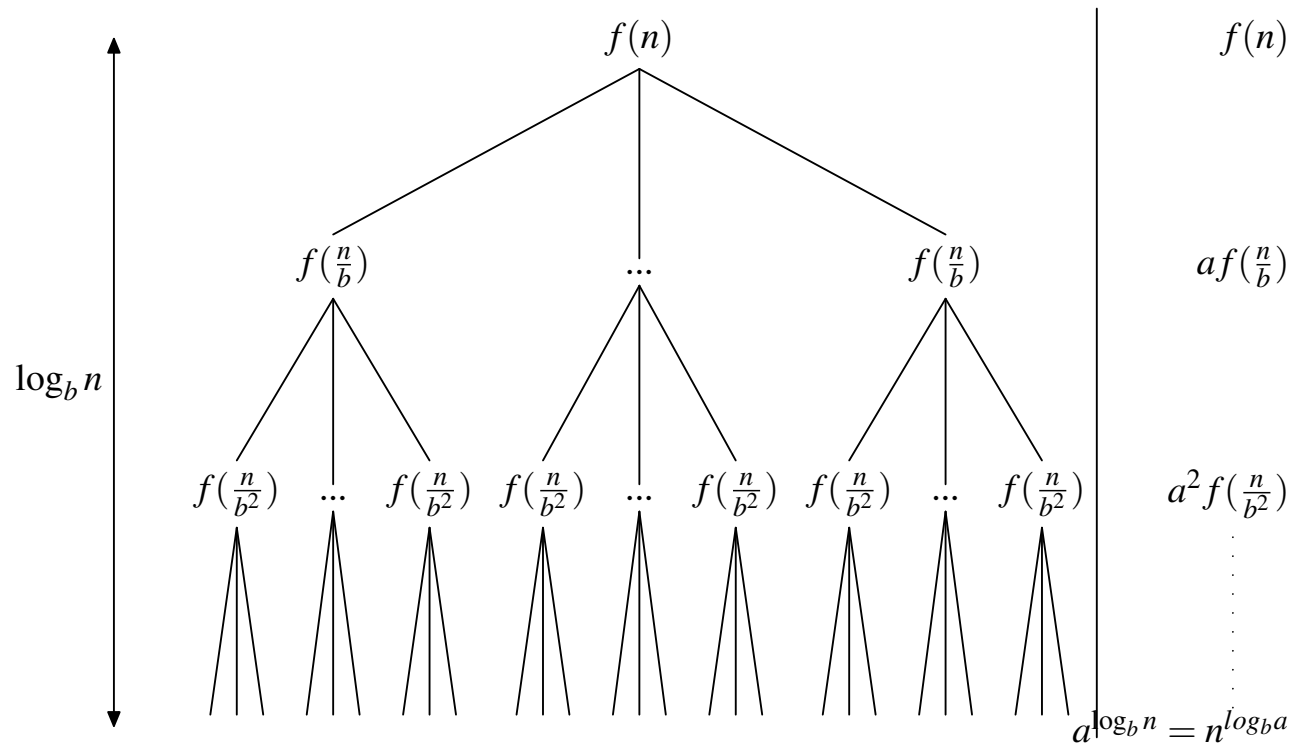
$$T_n = aT_{n/b} + f(n)$$



Case 1: $f(n)$ "larger" than $n^{\log_b a}$

Master Method

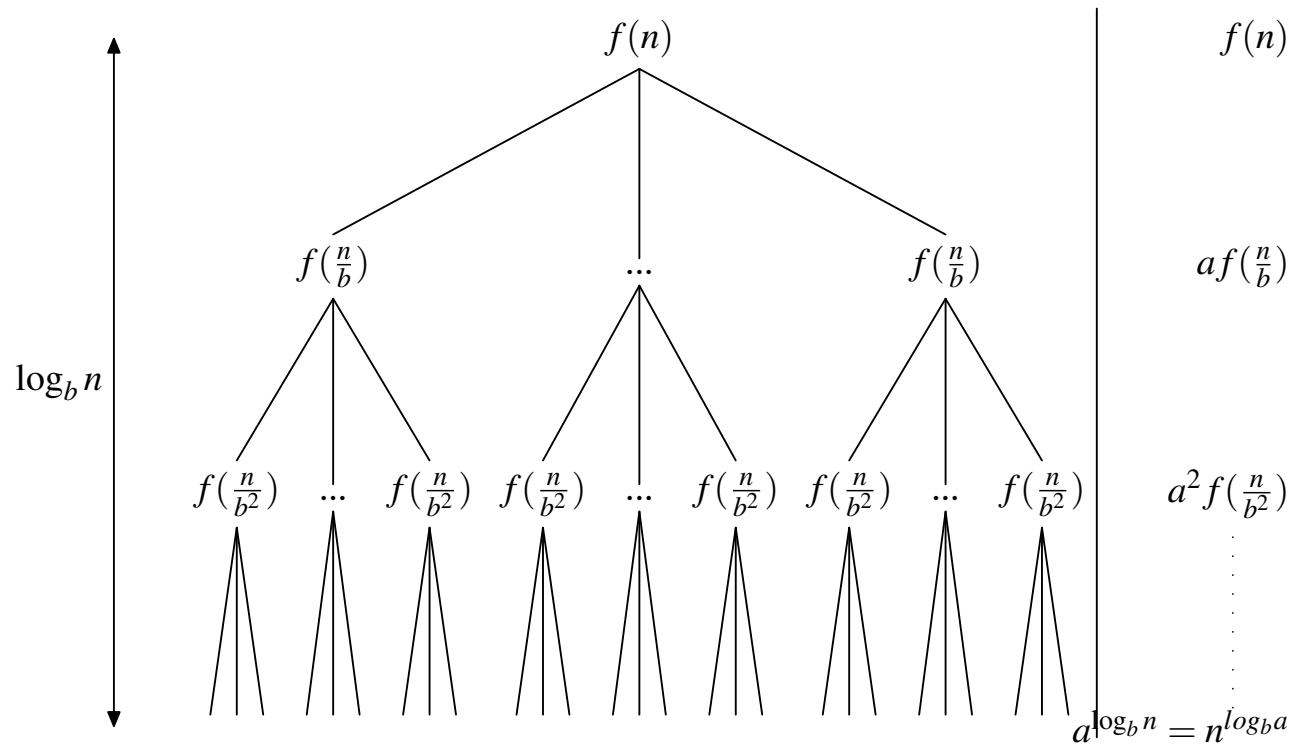
$$T_n = aT_{n/b} + f(n)$$



Case 2: $f(n)$ "smaller" than $n^{\log_b a}$

Master Method

$$T_n = aT_{n/b} + f(n)$$



Case 3: $f(n)$ "equal" $n^{\log_b a}$

Master Method Summary

Given

- $T_n = aT_{n/b} + f(n)$

Three general cases (in the long run):

- $f(n)$ grows faster than $n^{\log_b a}$
 - T_n grows proportional to $f(n)$
- $f(n)$ grows slower than $n^{\log_b a}$
 - T_n grows proportional to $n^{\log_b a}$
- $f(n)$ grows proportional to $n^{\log_b a}$
 - T_n grows proportional to $\log_b n \cdot n^{\log_b a}$

Master Method

Restrictions

- “Smaller” and “larger” mean polynomially smaller and larger
 - Must be some factor of n^ϵ different
- “Equal” means within a constant factor
- “Larger” also requires $f(n)$ is regular
 - As long as $f(n)$ is polynomial, it's regular

$f(n)$ may be in none of the three cases!

Examples of Master Method

$$T_n = 3T_{n/2} + n^2$$

$$T_n = 3T_{n/2} + n^{1.4}$$

$$T_n = 16T_{n/4} + 10n^2 + 3n + 5$$

Recurrence Relations

Linear homogenous recurrence relations with constant coefficients of order k

- Linear
- homogeneous
- constant coefficients
- order k

First-order example

$$T_n = 3T_{n-1}$$

Simple second-order example

$$T_n = 4T_{n-2}$$

- Guess that there's an exponential solution: $T_n = r^n$

Simple second-order example

$$T_n = 4T_{n-2}$$

$$T_0 = 1$$

$$T_1 = 3$$

- Find particular solution of recurrence given initial conditions
- $T_n = C2^n + D(-2)^n$

General Approach

Given a recurrence relation of form

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$ linear homogeneous with constant coefficients

Rewrite as powers:

- $\alpha^n = c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \dots + c_r \alpha^{n-r}$

Simplify

- $\alpha^r - c_1 \alpha^{r-1} - c_2 \alpha^{r-2} - \dots - c_r = 0$ characteristic equation

Solve for r distinct roots, $\alpha_1, \dots, \alpha_r$

Generate form of final solution ($a_n = \alpha_i^n$ is a solution to rec. relation)

- $a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_r \alpha_r^n$

Use initial conditions to create a set of simultaneous equations

- $a_0 = A_1 \alpha_1^0 + A_2 \alpha_2^0 + \dots + A_r \alpha_r^0$
- $\dots$
- $a_{r-1} = A_1 \alpha_1^{r-1} + A_2 \alpha_2^{r-1} + \dots + A_r \alpha_r^{r-1}$

Solve for $A_1, \dots, A_r$

Example

Find a closed form for the number of ways to make a stack of n chips using red, white and blue chips and so that no two red chips are together

- $a_n =$
- $a_0 =$
- $a_1 =$

Example

Create a recursive formula to specify how many ways to climb an n -stair staircase if each step covers either one or two stairsteps.

Example

A boy has n cents. Each day, he can buy one of:

- a candy bar (for 20 cents)
- a piece of gum (for 1 cent)
- an apple (for 10 cents)

How many ways are there he can buy items until the money is gone?

Example

How many n -digit ternary sequences without any occurrence of the subsequence 012?

Adjustment for Multiple Roots

Given a recurrence relation of form

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_r a_{n-r}$

...

Solve for r roots, $\alpha_1, \dots, \alpha_r$

If we have a root α_i of multiplicity 2, general form of equation is

- $a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots A_i \alpha_i^n + A_{i+1} n \alpha_i^n + \dots + A_r \alpha_r^n$

Example with Multiple Roots

Given

- $a_n = 4a_{n-1} - 4a_{n-2}$
- $a_0 = 1$
- $a_1 = 3$

Find a closed form for a_n