

CSE 21—Mathematics for Algorithm and System Analysis

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Day 11

Probability

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Monty Hall Problem

Game show (*Let's Make a Deal*) host Monty Hall does the following consistently at the end of each show:

- He shows three doors.
- One door contains the prize, two contain goats.
- Contestant chooses one door.
- Monty, **knowing what's behind each door**, opens one of the other doors to show a goat (he'll never show the prize).
- Monty offers the contestant the chance to switch doors.

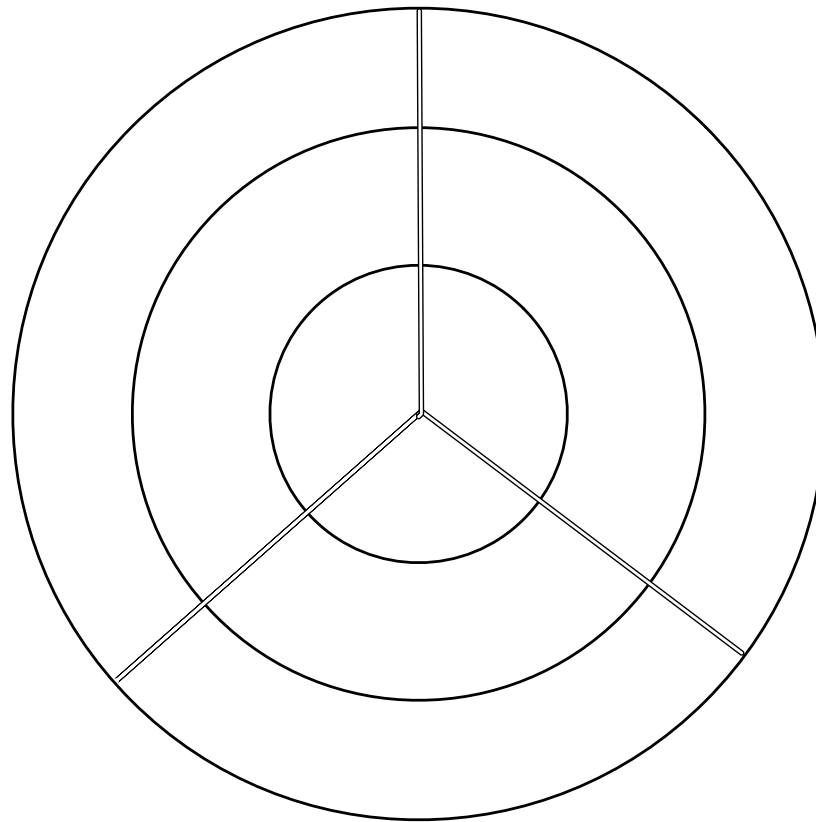
Should the contestant switch doors?

Monty Hall Sample Space

Inner ring: door containing the prize (equiprobable behind any door)

Middle ring: door the contestant choose (equiprobable door choice)

Outer ring: door Monty chooses (if has a choice, equiprobable)



Sample space diagram due to Adriano Garcia

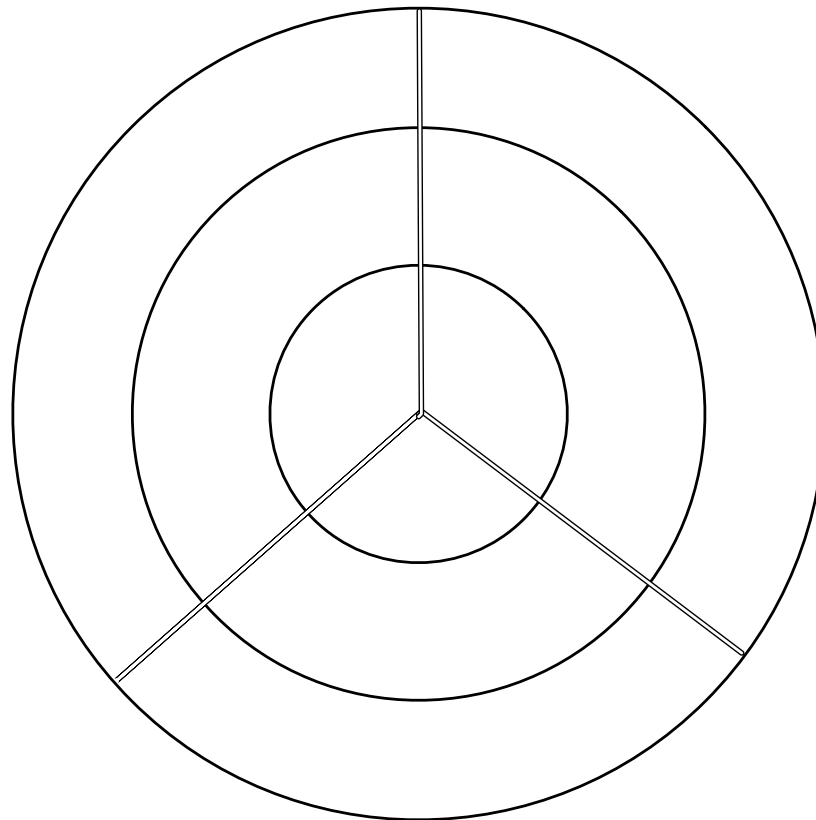
Monty Hall Incorrect Sample Space

Inner ring: door containing the prize (equiprobable behind any door)

Middle ring: door the contestant choose (equiprobable door choice)

Outer ring: door Monty chooses (if has a choice, equiprobable)

Monty could show the prize



Sample space diagram due to Adriano Garcia

Decision Tree

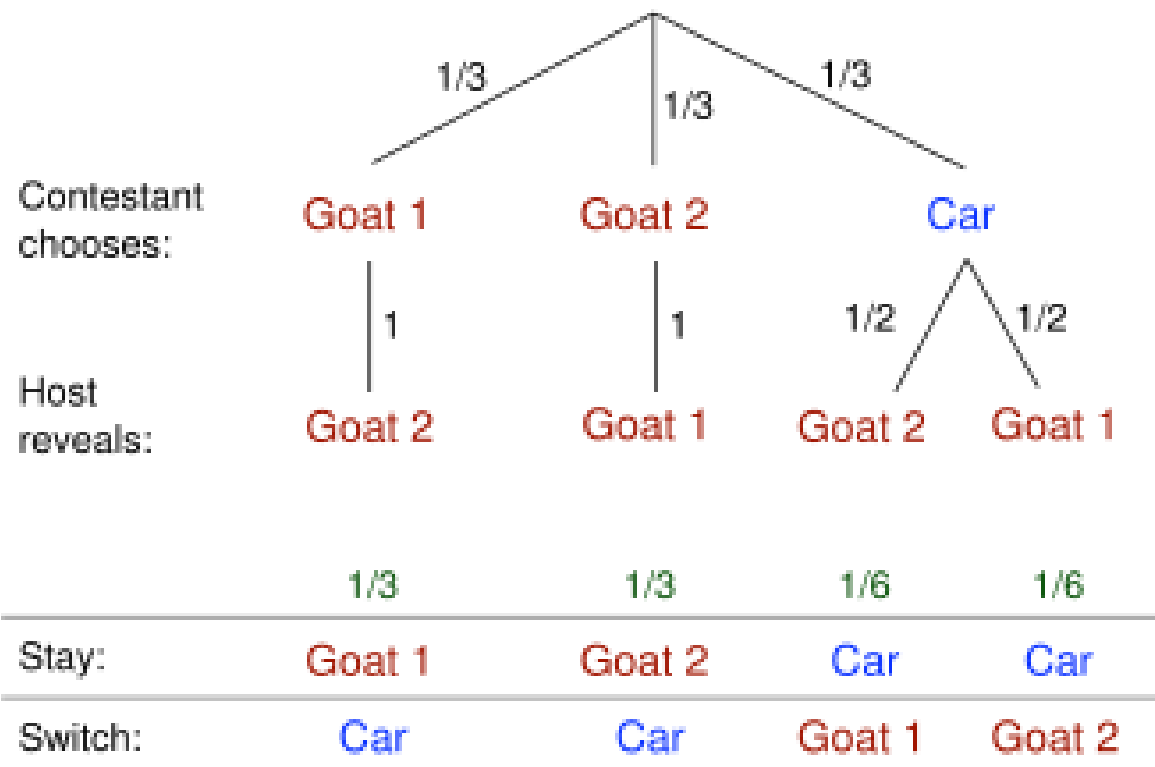
A tree that represents the result of a sequence of decisions

- Each internal node represents a particular decision to make
- Edges to children represent the result of the decision
- Leaves represent end result of the decision

Useful as a systematic way to generate all possibilities based on a sequence of decisions

A path from the root to a particular leaf shows the decisions that led to that end result

Monty Hall Decision Tree



Example

If we count the letters AEIOUY are vowels, how many five-letter words can we make, given the following rules:

- No vowels are ever adjacent
- There are never three consonants adjacent
- Adjacent consonants are always different

Conditional Probability

Given a probability space (S, Pr) , if A and B are events from S , then the conditional probability of B given A :

$$P(B|A) = P(A \cap B) / P(A)$$

Undefined when $P(A) = 0$

Two ways to think about it:

- 1) If we know that A occurred, then we have a new sample space, consisting of A . We must rescale P' so that $P'(A) = 1$. $P'(e) = P(e) / P(A)$. Probability that B occurs is $P'(B \cap A) = P(B \cap A) / P(A)$
- 2) Run experiment n times. Count how many times A occurred, and how many times $B \cap A$ occurred. We don't care about events in $B - (B \cap A)$. Proportion of events of interest out of total where A occurred is $P(B \cap A) / P(A)$

Properties of Conditional Probability

Let (S, P) be a probability space.

- $P(B|S) = P(B)$
- A and B are independent iff $P(B|A) = P(B)$
- $P(A|B) = P(B|A) P(A) / P(B)$
- $P(A_1 \cap \dots \cap A_n) = P(A_1) P(A_2|A_1) P(A_3|A_1 \cap A_2) \dots P(A_n|P(A_1 \cap A_2 \cap \dots \cap A_{n-1}))$

Bayes' Theorem Example

A test is used to screen for a disease. Three percent of people tested who don't have the disease test positive. Ten percent of people tested who *do* have the disease test negative. If the prevalence of the disease is 2% (about 2% of the subject population have it), what is the probability that a person who tests positive actually has the disease?

- Prob space: subject population with uniform distribution
- Event D: has disease
- Event T: tests positive

- Want to know: $P(D|T)$
 - $= P(T|D) \cdot P(D) / P(T)$ We know $P(T|D)$, $P(D)$, don't know $P(T)$
 - $P(T) = P(T \cap D) + P(T \cap D^c)$
 - $= P(T|D)P(D) + P(T|D^c)P(D^c)$
 - $= .9 \cdot .02 + .03 \cdot .98 = .047$

 - $P(D|T) = .9 \cdot .02 / .047 = .38$

Reanalysis using Decision Tree

Example

We are given an urn with one red ball and one white ball. A fair die is thrown.

- If the number is 1: add 1 red ball and 2 white balls
- If the number is 2 or 3: add 2 red balls and 3 white balls
- If the number is 4, 5, or 6, add 3 red balls and 4 white balls

Then, a ball is selected uniformly at random from the urn.

- Given that the ball chosen was red, what is the probability that the roll was a 2 or 3?

Example

You have two coins: one normal, one two-headed. Select a coin randomly and toss it. If it comes up heads, switch the coin you're holding. Now, toss the coin you're holding.

- What is the probability that your second toss is a H?
- Given that the final toss was heads, what is the probability that the first coin was normal?

Yet Another Decision Tree Example

Team A plays Team B in a tournament

- The probability of team A winning any game is $1/3$.
- If any team wins 2 games in a row, that team is the champ.
- At most four games are played
 - If no champion, it's a draw

Given that the tournament lasts more than 2 games, what is the probability that A is the winner?

Example

In a certain college 4% of the men and 1% of the women are over 6 ft. tall. 60% of the students are women. A randomly selected student is over 6 ft. tall. What's the probability the student is a woman?

Independence

A and B are independent events (in (S, p)) if:

- $P(A \cap B) = P(A)P(B)$

Independence Example

Consider the following events for a family with children:

- $A = \{\text{children of both sexes}\}$ $B = \{\text{at most one boy}\}$

Show that A and B are independent events if a family has 3 children

Show that A and B are dependent events if a family has 2 children

Example

Let $S = \{a, b, c, d\}$ with uniform probability. Let:

- $A = \{a, d\}$ $B = \{b, d\}$, $C = \{c, d\}$

Show that A, B, C are pairwise independent

Show that A, B, C , are not independent

Expected value (Expectation)

Expected value = sum of $P(e)$ *value of outcome e

Flip a coin. Heads, you pay \$5, tails, you win \$6. What is the expected value of amount you win?

Expectation using decision tree

Buy parking pass/Don't buy parking pass

Expectation Paradox

Flip a coin until you get a tail. You get paid $\$2^{\text{number_of_heads}}$.
What is your expected value?