

Knapsack Problem

$$\text{Maximize } V = \sum_{i=1}^n v_i x_i$$

$$\text{Subject to } W = \sum_{i=1}^n w_i x_i \leq b$$

$$x_i \geq 0, \text{ integers}$$

b : boundary (carry-capacity)

Best item $\frac{v_1}{w_1} > \frac{v_j}{w_j}$

Optimal solution is periodic

when b is sufficiently large

Example

$$\text{Max } V = 8x_1 + 3x_2 + 17x_3$$

$$\text{Sub } W = 8x_1 + 5x_2 + 18x_3 \leq b$$

$$x_i \geq 0, \text{ integers}$$

Goal: To find the exact points where the optimal periodic solution start

Example

Boundary	Optimal Solution				
b	x_1	x_2	x_3	W	V
24	3	0	0	24	24
25	3	0	0	24	24
26	1	0	1	26	25
27	1	0	1	26	25
28	1	0	1	26	25
29	3	1	0	29	27
30	3	1	0	29	27
31	1	1	1	31	28
32	4	0	0	32	32
33	4	0	0	32	32
34	2	0	1	34	33
35	2	0	1	34	33
36	0	0	2	36	34
37	4	1	0	37	35
38	4	1	0	37	35
39	2	1	1	39	36
40	5	0	0	40	40
41	5	0	0	40	40
42	3	0	1	42	41
43	3	0	1	42	41
44	1	0	2	44	42
45	5	1	0	45	43
46	5	1	0	45	43
47	3	1	1	47	44

Gain

Gain of a solution to a boundary b :

The value of a solution minus the value of the best-item-only solution

$$G_b = V - v_1 \left\lfloor \frac{b}{w_1} \right\rfloor$$

Example

Boundary	Optimal Solution					
b	x_1	x_2	x_3	W	V	g
24	3	0	0	24	24	0
25	3	0	0	24	24	0
26	1	0	1	26	25	1
27	1	0	1	26	25	1
28	1	0	1	26	25	1
29	3	1	0	29	27	3
30	3	1	0	29	27	3
31	1	1	1	31	28	4
32	4	0	0	32	32	0
33	4	0	0	32	32	0
34	2	0	1	34	33	1
35	2	0	1	34	33	1
36	0	0	2	36	34	2
37	4	1	0	37	35	3
38	4	1	0	37	35	3
39	2	1	1	39	36	4
40	5	0	0	40	40	0
41	5	0	0	40	40	0
42	3	0	1	42	41	1
43	3	0	1	42	41	1
44	1	0	2	44	42	2
45	5	1	0	45	43	3
46	5	1	0	45	43	3
47	3	1	1	47	44	4

Threads

Partition all values of b into w_1 class

$b \pmod{w_1} = 0$ Thread 0

$b \pmod{w_1} = 1$ Thread 1

$\vdots$

$b \pmod{w_1} = 7$ Thread 7

Preferred Optimal Solution

The optimal solution which uses best item most

We use a table where every thread is a

column, with many cells

b
G

Example

Threads							
0	1	2	3	4	5	6	7
$b = 0$ $g = 0$	$b = 1$ $g = 0$	$b = 2$ $g = 0$	$b = 3$ $g = 0$	$b = 4$ $g = 0$	$b = 5$ $g = 3$	$b = 6$ $g = 3$	$b = 7$ $g = 3$
$b = 8$ $g = 0$	$b = 9$ $g = 0$	$b = 10$ $g = 0$	$b = 11$ $g = 0$	$b = 12$ $g = 0$	$b = 13$ $g = 3$	$b = 14$ $g = 3$	$b = 15$ $g = 3$
$b = 16$ $g = 0$	$b = 17$ $g = 0$	$b = 18$ $g = 1$	$b = 19$ $g = 1$	$b = 20$ $g = 1$	$b = 21$ $g = 3$	$b = 22$ $g = 3$	$b = 23$ $g = 4$
$b = 24$ $g = 0$	$b = 25$ $g = 0$	$b = 26$ $g = 1$	$b = 27$ $g = 1$	$b = 28$ $g = 1$	$b = 29$ $g = 3$	$b = 30$ $g = 3$	$b = 31$ $g = 4$
$b = 32$ $g = 0$	$b = 33$ $g = 0$	$b = 34$ $g = 1$	$b = 35$ $g = 1$	$b = 36$ $g = 2$	$b = 37$ $g = 3$	$b = 38$ $g = 3$	$b = 39$ $g = 4$
$b = 40$ $g = 0$	$b = 41$ $g = 0$	$b = 42$ $g = 1$	$b = 43$ $g = 1$	$b = 44$ $g = 2$	$b = 45$ $g = 3$	$b = 46$ $g = 3$	$b = 47$ $g = 4$
$b = 48$ $g = 0$	$b = 49$ $g = 0$	$b = 50$ $g = 1$	$b = 51$ $g = 1$	$b = 52$ $g = 2$	$b = 53$ $g = 3$	$b = 54$ $g = 3$	$b = 55$ $g = 4$
$b = 56$ $g = 0$	$b = 57$ $g = 0$	$b = 58$ $g = 1$	$b = 59$ $g = 1$	$b = 60$ $g = 2$	$b = 61$ $g = 3$	$b = 62$ $g = 3$	$b = 63$ $g = 4$

Special Boundary

The cell which has an optimal gain
larger than the gain of the cell above it

Example:

b=12 g=0	
b=20 g=1	is a special boundary
b=28 g=1	
b=30 g=2	is a special boundary

Lemma 1

Lemma 1: Let G_b and G_s be the gains of two adjacent cells in a column, then $G_b \geq G_s$ when $b_b = b_s + w_1$.

If we use the solution of b_s as a subsolution of b_b , then

$$G_b = G_s$$

Proof

$$\begin{aligned} G_b &= V_b - v_1 \left\lfloor \frac{b_b}{w_1} \right\rfloor \\ &= (V_s + v_1) - v_1 \left\lfloor \frac{b_s + w_1}{w_1} \right\rfloor \\ &= V_s + v_1 - v_1 - v_1 \left\lfloor \frac{b_s}{w_1} \right\rfloor \\ &= V_s - v_1 \left\lfloor \frac{b_s}{w_1} \right\rfloor \\ &= G_s \end{aligned}$$

Example

$$b_b = 36, b_s = 28$$

$$\text{then } G_b = 2, G_s = 1$$

$$\text{The solution of } b_s = 28 = w_1 + w_3$$

If we let

$$b_b = 36 = (w_1 + w_3) + w_1$$

$$\text{then } G_b = 1 = G_s$$

Lemma 2 & 3

Lemma 2: Preferred optimal solutions to special boundaries have $x_1 = 0$.

Proof: If $x_1 \neq 0$, then the two gains are the same

Lemma 3: Preferred optimal solutions to non-special boundaries have $x_1 \geq 1$

Lemma 4

Lemma 4: Optimal gains in any thread
are non-negative

Proof: Best-item-only solution is a
feasible solution

Lemma 5

Lemma 5: Optimal gains in a column are monotonically increasing as b increases

Proof: By Lemma 1,

$$G_b = G_s \quad (b_b = b_s + w_1)$$

When b_b contains b_s subsolution, the optimal solution is at least as good as the feasible solution

Lemma 6

Lemma 6: Gains are smaller than v_1

By def. $V = G + v_1 \left\lfloor \frac{b}{w_1} \right\rfloor$

If $G = v_1$, then

$$V_b = v_1 + v_1 \left\lfloor \frac{b}{w_1} \right\rfloor = v_1 \left\lfloor \frac{b + w_1}{w_1} \right\rfloor$$

$\Rightarrow$ the density of the total knapsack is
better than best item type.

Lemma 7

Lemma 7: Within every thread, $\exists$ a cell with Max gain which remains constant as b increases

Proof: Each increase is an integer, with upper bound v_1 (Lemma 6)

e.g. The max gain cell in Thread 4 is $b = 36$

Theorem 1

TH 1: Given $b_b = b_s + w_k$ in two thread t_b & t_s , and the solution of b_b contains the solution of b_s as a subsolution plus one copy of w_k . Then the two gains G_b and G_s are related by

$$G_b = G_s + v_k - v_1 \left\lfloor \frac{t_s + w_k}{w_1} \right\rfloor$$

Theorem 1

The following identity is used in the proof of Theorem 1.

$$\begin{aligned}\left\lfloor \frac{b_s + w_k}{w_1} \right\rfloor &= \left\lfloor \frac{b_s \pmod{w_1} + w_1 \left\lfloor \frac{b_s}{w_1} \right\rfloor + w_k}{w_1} \right\rfloor \\ &= \left\lfloor \frac{b_s + w_k}{w_1} \right\rfloor + \left\lfloor \frac{b_s}{w_1} \right\rfloor\end{aligned}$$

Proof of Theorem 1

$$\begin{aligned} G_b &= V_b - v_1 \left\lfloor \frac{b_b}{w_1} \right\rfloor \\ &= V_s + v_k - v_1 \left\lfloor \frac{b_s + w_k}{w_1} \right\rfloor \\ &= V_s + v_k - v_1 \left(\left\lfloor \frac{t_s + w_k}{w_1} \right\rfloor + \left\lfloor \frac{b_s}{w_1} \right\rfloor \right) \\ &= V_s + v_k - v_1 \left\lfloor \frac{b_s}{w_1} \right\rfloor - v_1 \left\lfloor \frac{t_s + w_k}{w_1} \right\rfloor \\ &= G_s + v_k - v_1 \left\lfloor \frac{t_s + w_k}{w_1} \right\rfloor \end{aligned}$$

Theorem 2

If a preferred optimal solution to a
thread critical boundary b_b is optimal,
then its subsolution to $b_s = b_b - w_k$ is
optimal

Proof: Principle of optimality

High-level Algorithm

- Partition b into w_1 Threads
- Introduce each type of items starting with the best type
- In each Thread, keep the cell with Max Gain and Min Boundary

Case 1

$$\gcd(w_1, w_2) = 1$$

$$= \gcd(8, 5) = 1$$

Visit

$T_0, T_5, T_2, T_7, T_4, T_1, T_6, T_3$

T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
$b = 0$		$b = 0$			$b = 5$		$b = 5$
$g = 0$		$g = 0$			$g = 3$		$g = 3$

$$\begin{pmatrix} 8 \\ -8 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Case 2

$$\gcd(w_1, w_3)$$

$$\gcd(8, 18) \neq 1$$

Visit

T_0, T_2, T_4, T_6 in one chain

T_1, T_3, T_5, T_7 in one chain

double-loop technique

Complexity

Step 1. Find best item $O(n)$

Step 2. Partition into w_1 Thread $O(w_1)$

Step 3. For each type of items visit all
Threads $O(nw_1)$

Time Complexity $O(nw_1)$

Example

$$\text{Max } 7x_1 + 699,998x_2$$

$$\text{Sub } 5x_1 + 499,999x_2 \leq b$$

$$x_i \geq 0, \text{ integers}$$

	T_0	T_1	T_2	T_3	T_4
$\left(\frac{7}{5}\right)$	$b = 0$ $G = 0$	1 0	2 0	3 0	4 0
$\left(\frac{v_2}{w_2}\right)$	$b = 0$ $G = 0$	$b = 1$ $G = 0$	$b = 3w_2$ $G = 1$	$b = 2w_2$ $G = 3$	$b = w_2$ $G = 5$

Example

$b = 2,000,002$ in Thread 2

$$\text{Gain} \quad G = V - v_1 \left\lfloor \frac{b}{w_1} \right\rfloor$$

$$V = v_1 \left\lfloor \frac{b}{w_1} \right\rfloor + G$$

$$V = 7 \left\lfloor \frac{2,000,002}{5} \right\rfloor + 1$$

$$= 2,800,001.$$