Prim's Algorithm is Correct

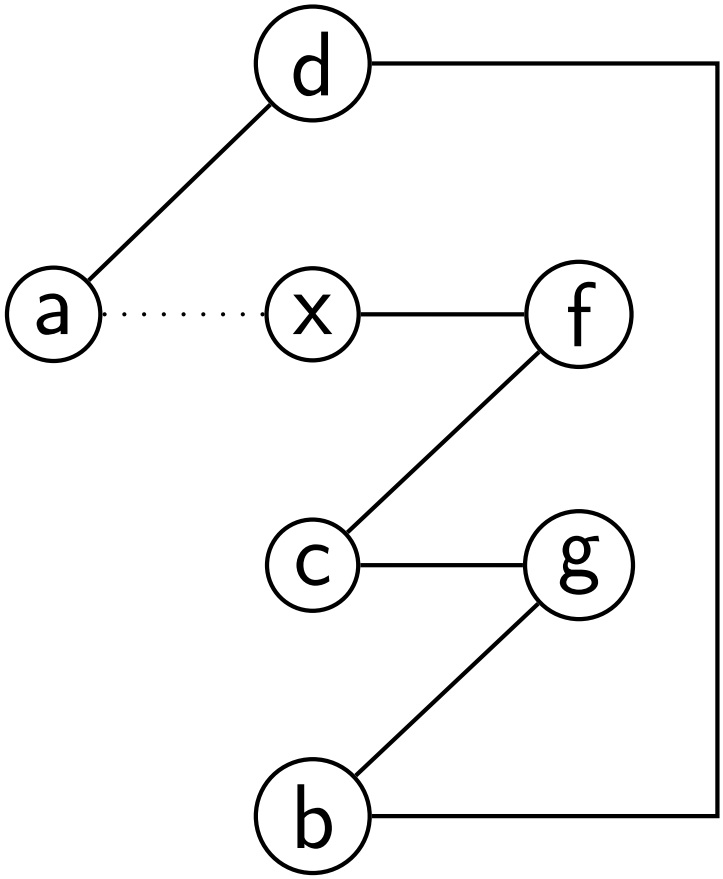
Lemma 1:

Let e_{ax} be the shortest edge among all edges adjacent to an arbitrary node V_a. Then there exists a MST T^* which includes the edge e_{ax}.

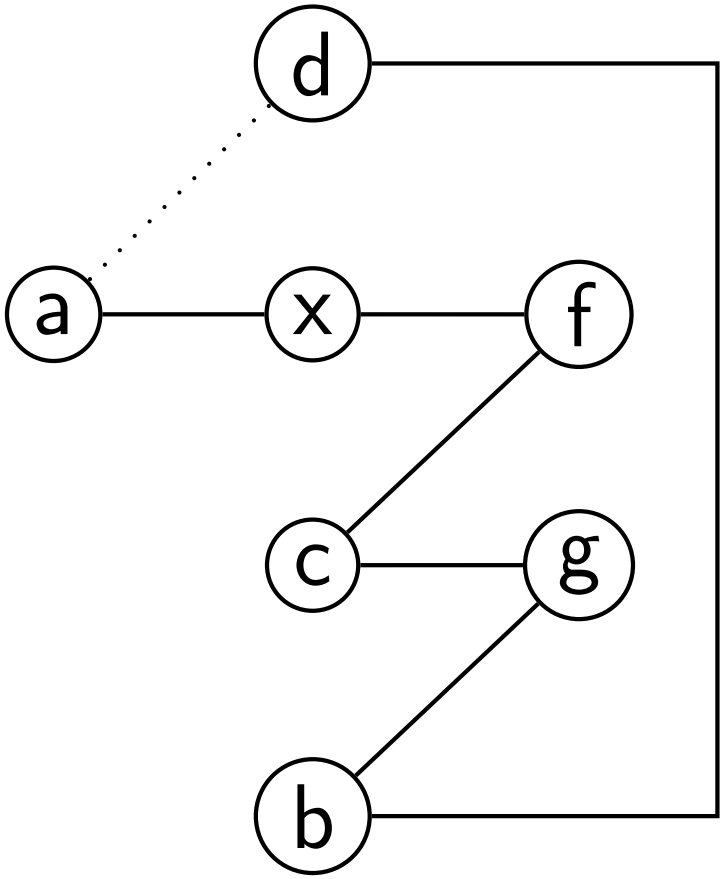
Prim's Algorithm is Correct

Proof (by contradiction): Suppose that e_{ax} is not, but some other edge e_{ad} is. Since the tree is a spanning tree, $\exists$ a path from V_x to V_d, so replace e_{ad} by e_{ax} to get a cheaper spanning tree.

Lemma 1 Example



Lemma 1 Example



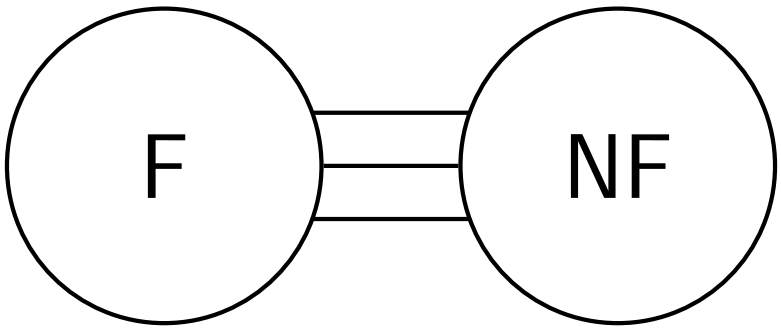
Prim's Algorithm is Correct

Lemma 2:

If a subset of edges forming a subtree F is known to be part of a MST, then there exists a MST containing F and the minimum cost edge from F to $N - F$.

Conjecture

Partition the nodes in two parts F and $N - F$ arbitrarily and construct the MST on both sets. Then select the edge of min cost connecting the two parts to obtain the MST.



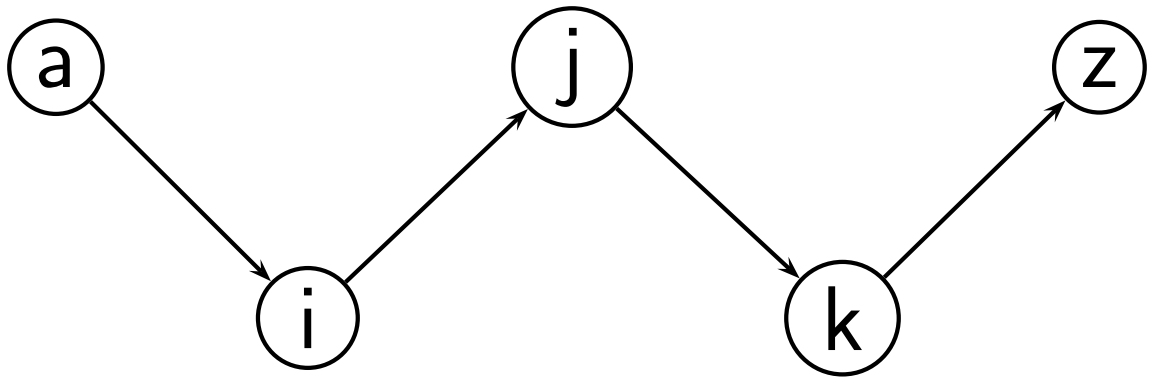
Multinomial Shortest Paths

$d_{ij} \neq d_{ji}$ in general

$d_{ij} \gtrless 0$ but no neg. cycles

$d_{ij} + d_{jk} \not\geq d_{ik}$ in general

Finding the Shortest Distance



A subpath of a shortest path

is itself a shortest path.

Basic arc (vs. yellow arc)

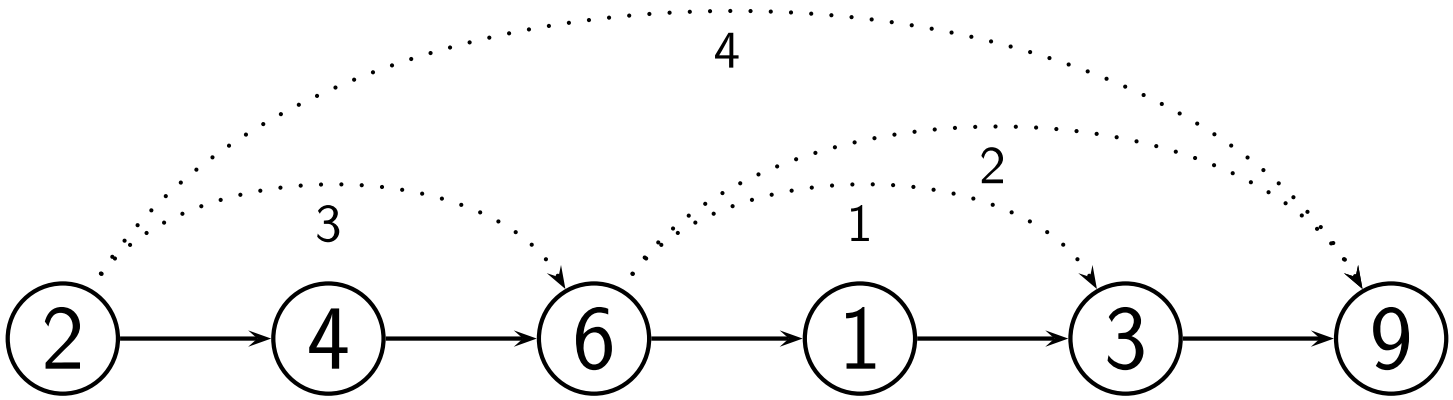
Triple Operation

$$d_{ik} \leftarrow \min(d_{ik}, d_{ij} + d_{jk})$$

$$\begin{array}{lll} \text{for} & j = 1, & \text{all } i, k \neq j \\ & j = 2, & \text{all } i, k \neq j \\ & \vdots & \\ & j = n, & \text{all } i, k \neq j \end{array}$$

Example

If for an arbitrary index we can find the shortest path, it is a proof



Question: In the case of a tie, which path will be obtained?

Finding the Actual Paths

We found the correct shortest distances

but not the intermediate nodes

	1	2	3	4	5
1	1	2	3	4	5
2	1	2	3	4	5
3	1	2	3	4	5
4	1	2	3	4	5
5	1	2	3	4	5

Start with $P_{ik} = k$,

then update as d_{ik} changes

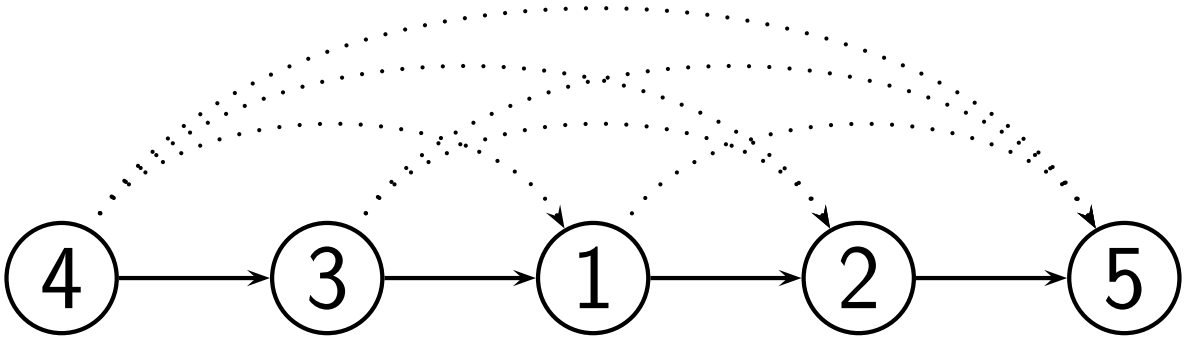
Update Rule

$$P_{ik} = \begin{cases} j & \text{if } d_{ij} + d_{jk} < d_{ik} \\ \text{same} & \text{if } d_{ij} + d_{jk} \geq d_{ik} \end{cases}$$

$$P_{ik} = \begin{cases} P_{ij} & \text{if } d_{ij} + d_{jk} < d_{ik} \\ \text{same} & \text{if } d_{ij} + d_{jk} \geq d_{ik} \end{cases}$$

This gives you the first intermediate node

Example



When $j = 1$, $P_{32} \leftarrow P_{31} = 1$

When $j = 1$, $P_{15} \leftarrow P_{12} = 2$

When $j = 1$, $P_{45} \leftarrow P_{43} = 3$

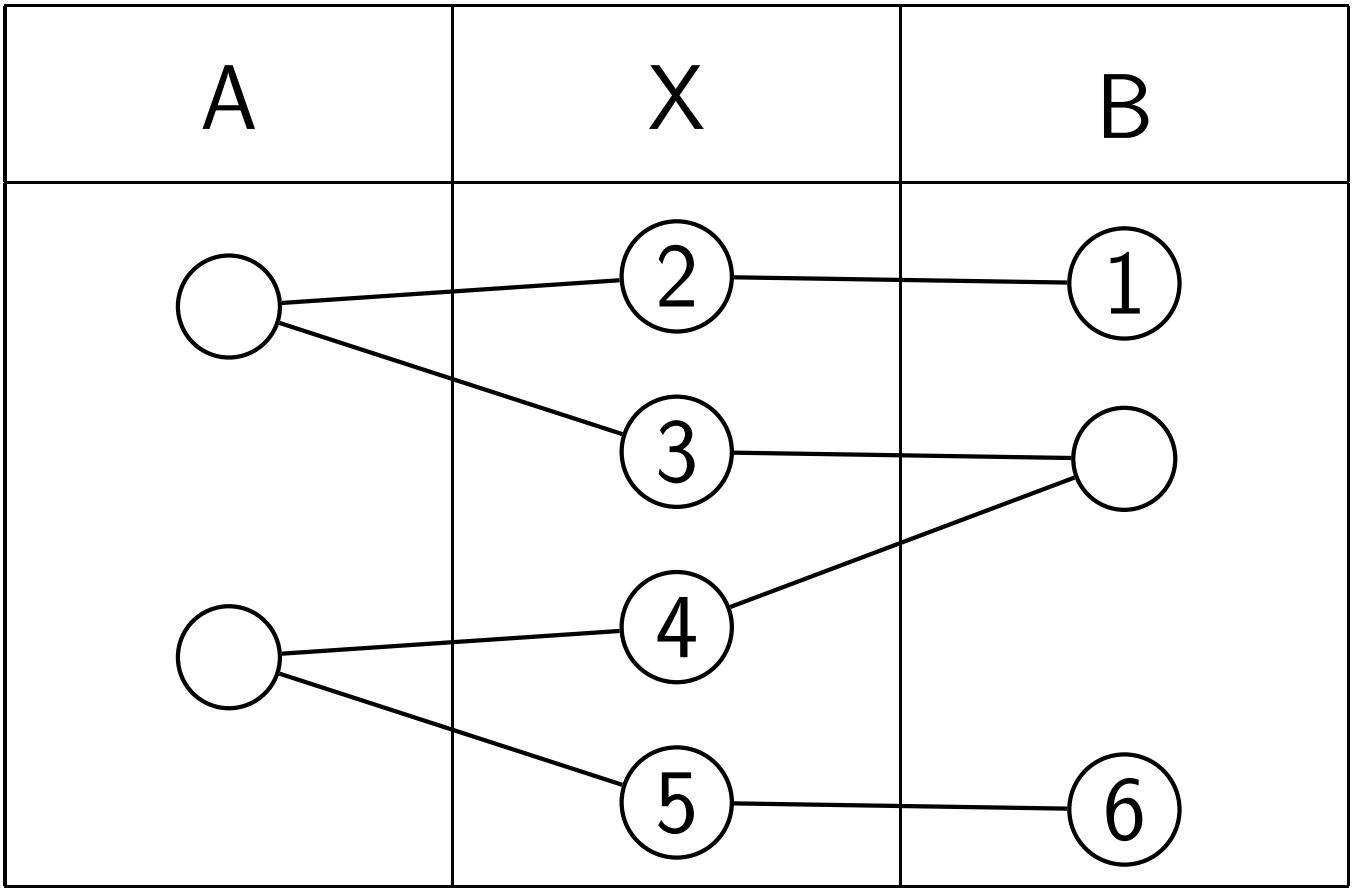
First intermediate node $= 3$

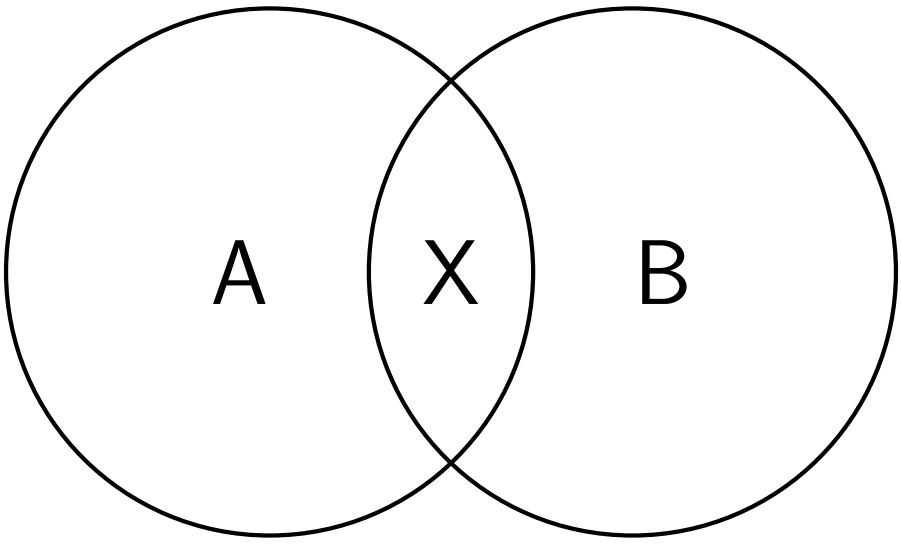
First intermediate node $= 1$

First intermediate node $= 2$

First intermediate node $= 5$

Decomposition of the Triple Operation





Other Cost Functions

$$d_{ik} \leftarrow \min(d_{ik}, \max(d_{ij}, d_{jk}))$$

$$d_{ik} \leftarrow \max(d_{ik}, \min(d_{ij}, d_{jk}))$$

Negative Cycle?