
Greedy Algorithms

Dynamic Programming

Backtracking

Implicitly Search all Possibilities

Find all positive integer solutions
satisfying

$$3x_1 + 4x_2 + x_3 \leq 10$$

$$1 \leq x_i \leq 3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \dots, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

A vector $[x_1, x_2, \dots, x_n]$ is
lexicographically positive if the top/left
most nonzero component is positive.

E.g. $[0, 0, 1, -9]$ is positive,
 $[0, -1, 9, 2]$ is negative,
and $[0, 0, 1, -9] \geq [0, -1, 9, 2]$

Start with the lexicographically smallest vector,
try all vectors which satisfy the constraints.

$$3x_1 + 4x_2 + x_3 \leq 10$$

$$3x_1 + 4x_2 - x'_3 \leq 10$$

$$1 \leq x_i \leq 3$$

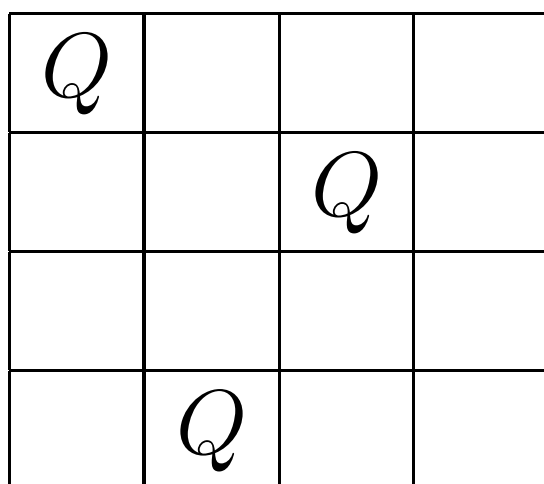
$$[1, 1, 1] \quad [1, 2, 1]?$$

$$[1, 1, 2] \quad [1, 2, 2]?$$

$$[1, 1, 3] \quad [1, 2, 3]?$$

If $3x_1 + x_2 \geq 10$ already, no need to try $x_3 = ?$

Put 4 Q on a 4×4 chess board so that no pair of queens can attack each other.



Configuration $\Leftrightarrow$ Vectors

$x_1 = 1, 2, 3, 4$ means Q in the 1st row
and 1st, 2nd, 3rd, 4th column.

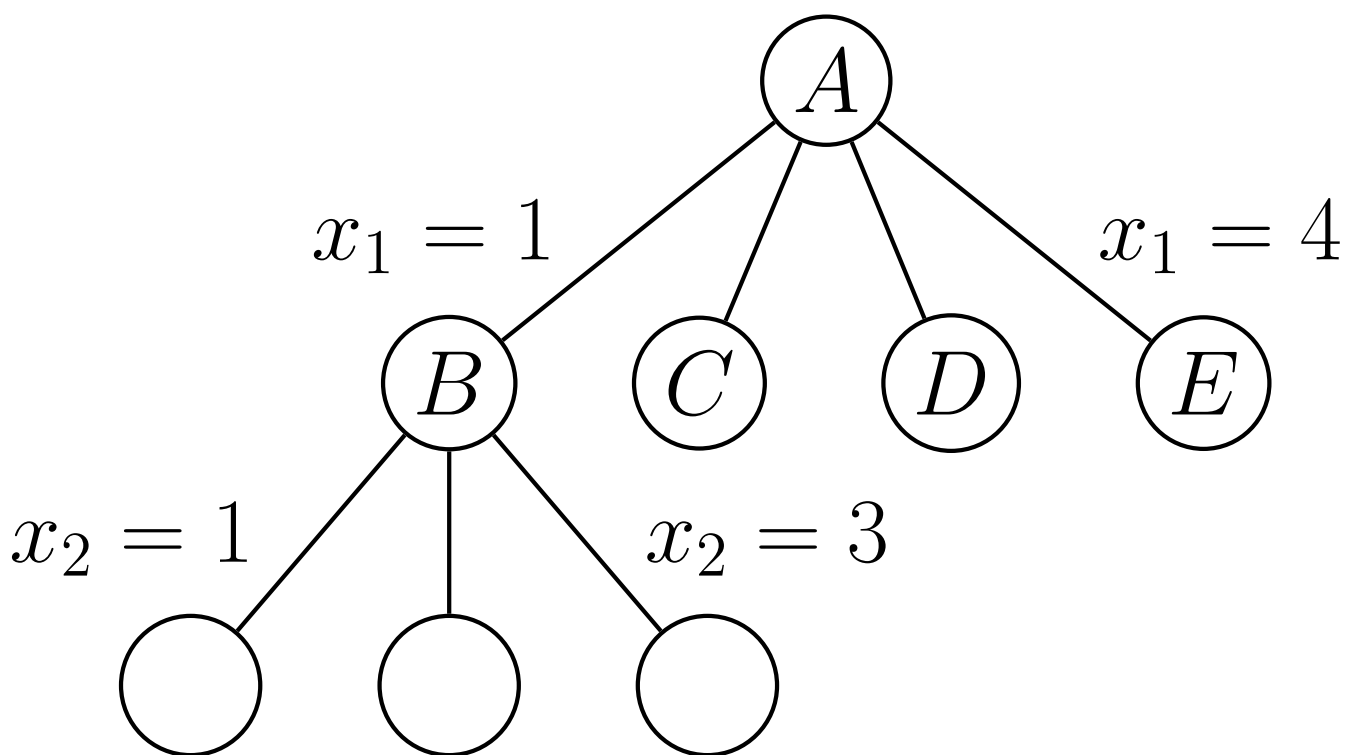
$x_2 = 3$ means Q in 2nd row and 3rd
column.

Q			
		Q	
	Q		

$$\begin{bmatrix} x_1 = 1 \\ x_2 = 3 \\ x_3 = 0 \\ x_4 = 2 \end{bmatrix}$$

$[1, 1, 1, 1], [1, 1, 1, 2], \dots, [4, 4, 4, 4]$.

Decision Tree



$P_4(x_1, x_2, x_3, x_4)$ property of the vector
 $[x_1, x_2, x_3, x_4]$ that says that no Q 's can
attack each other.

$P_2(x_1, x_2)$ property of the vector
 $[x_1, x_2, -, -]$.

$P_4(x_1, x_2, x_3, x_4)$ true
 $\implies P_3(x_1, x_2, x_3)$ true.

P_3 false $\implies P_4$ false.

A circular node has some component
unassigned.

A vector is a feasible vector if all
constraints are satisfied.

BACKTRACK
is
DFS in a Ordered Tree

Q			
		Q	

$$x_1 = 1$$

$$x_2 = 3$$

Q			
			Q
	Q		

$$x_1 = 1$$

$$x_2 = 4$$

$$x_3 = 2$$

	Q		
			Q
Q			
		Q	

$$x_1 = 2$$

$$x_2 = 4$$

$$x_3 = 1$$

$$x_4 = 3$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}$$

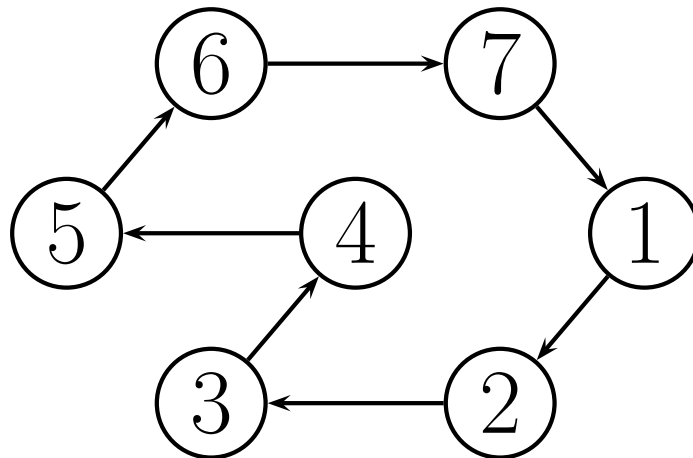
Branch and Bound

In backtrack, a partial vector and its children are excluded if the partial vector already violates requirements.

$$\textit{cost}(x_1, x_2) \leq \textit{cost}(x_1, x_2, x_3)$$

cost of father $\leq$ *cost* of son

TSP



Solve Knapsack

$$\max \sum v_i x_i$$

$$\text{sub } \sum w_i x_i \leq b$$

$$x_i \geq 0, \text{ integers}$$

$$\frac{v_i}{w_i} \geq \frac{v_{i+1}}{w_{i+1}}$$

$$\max 9x_1 + 5x_2 + 3x_3 + x_4$$

$$\text{sub } 7x_1 + 4x_2 + 3x_3 + 2x_4 \leq 10$$

$$x_j \geq 0, \text{ integers}$$

Do not branch if weight $\geq$ capacity.

How can we exclude a father and its children.

$$\textit{value}(x_1, x_2, x_3, \dots) = \\ v_1x_1 + v_2x_2 + v_3x_3 + \dots$$

$$\sum_{i=1}^k v_i x_i + \left[b - \sum_{i=1}^k w_i x_i \right] \frac{v_{i+1}}{w_{i+1}}$$

if $b - \sum_{i=1}^k w_i x_i \geq w_j$ for some
 $j \geq i + 1$

or $\sum_{i=1}^k v_i x_i$ if $b - \sum_{i=1}^k w_i x_i < w_j$ for
all $j \geq k$