Day 2.5

Discussion session

Neil Rhodes

UC San Diego

Overview

- Proofs
- $O(n)$
- Recurrence relations

Types of Proofs

Induction A base case is proved and an induction rule is used to prove a series of other cases.

By contradiction To prove a, assume $\neg a$. Show that leads to a contradiction

Contrapositive To prove $a \longrightarrow b$, prove $\neg b \longrightarrow \neg a$

By adversary Adversary provides value for all universal quantifiers

Weak inductive proof

- Inductive step: assume true for $n-1$, show true for n.
- Example: Prove$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - base case: $n = 1 : 1(1+1)/2 = 1$
  - inductive step: assume true for $n-1$ ($\sum_{i=1}^{n-1} i = \frac{n-1(n-1+1)}{2}$), show true for n

$$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i = n + \frac{(n-1)n}{2}$$

$$= \frac{2n + n(n-1)}{2} = \frac{n(n+1)}{2}$$

Inductive proof

- Inductive step: assume true for $1..n-1$, show true for n
- Definitions:
  - Binary tree: empty or: left binary tree + right binary tree + root node
  - $depth(node)$ = length of path from node to root
  - height of tree = maximum depth of its nodes (-1 for empty tree)

Inductive proof

- Example: Prove that if a binary tree T has height h, then T has fewer than 2^{h+1} nodes
- Base case (on height) $h = -1$: T has 0 nodes which is less than $2^{-1+1} = 2^0 = 1$ nodes
- Inductive step: Assume that trees of height of $h - 1$ or less have at most $2^{h-1} - 1$ nodes. Show true for an arbitrary tree, T, of height h.
  Nodes in left subtree and right subtree have depth 1 less than in T. Thus, each subtree has height $\leq h - 1$. By inductive step, each has at most $2^h - 1$ nodes. Total nodes in tree $\leq 2^h - 1 + 2^h - 1 + 1 = 2^{h+1} - 1$. QED.

O(n)

- Prove $f(n) = 3n^3 - 10n^2 + n - 10 = O(n^3)$
- Need to come up with c and n_0 such that $0 \leq f(n) \leq c \cdot n^3$
- $f(n) \geq 0$ for $n \geq 4$. So, let $n_0 = 4$
- $3n^3 - 10n^2 + n - 10 < 3n^3 + n$
- $3n^3 + n \leq 3n^3 + n^3 = 4n^3$
- So, let $c = 4$
- We've shown that $\forall n \geq 4, 0 \leq f(n) \leq 4n^3$

O(n)

- Prove $f(n) = 3n^3 - 10n^2 + n - 10 \neq O(n^2)$
- Assume, for the sake of contradiction, that $f(n) = O(n^2)$.
- Thus,$\exists c$ and n_0 such that $\forall n \geq n_0, 0 \leq f(n) \leq 4n^3$
- So, $\forall n \geq n_0, 3n^3 - 10n^2 + n - 10 \leq c \cdot n^2$
- Therefore, $\forall n \geq n_0, 3n^3 \leq (c+10)n^2 - n + 10$
- Dividing by 3n, $n \leq \frac{c+10}{3} - \frac{1}{3n} + \frac{10}{3n^2} \leq (c+10) + 10 \leq c + 20$
- Let $n = max(n_0, c+21)$. Then, $c + 21 \leq n \leq c + 20$, , a contradiction.
- Therefore $f(n) \neq O(n^2)$

Recurrence relations

- Show $T(n) = T(n-1) + 2$ is $O(n)$ (Unstated assumption: $T(1)$ = some constant, k)
- Show $T(n) \leq cn$ for some $c > 0$
- Inductive step: Assume $T(n-1) \leq c(n-1)$ show $T(n) \leq cn$
- $T(n) \leq T(n-1) + 2 = c(n-1) + 2 = cn - c + 2$
- $T(n) \leq cn$ (if $c \geq 2$)
- Base case: $T(1) \leq c \cdot 1$. True if $c \geq k$.
- Thus, let $c = max(2,k), n_0 = 1$

Recurrence relations

- Show $T(n) = 5T(\lfloor \frac{n}{5} \rfloor) + 6$ is $O(n \log n)$ (Unstated assumption: $T(1)$ = some constant, k)
- Show $T(n) \leq c(n \log n)$ for some $c > 0$
- Inductive step: Assume $T(\lfloor \frac{n}{5} \rfloor) \leq c(\lfloor \frac{n}{5} \rfloor \log(\lfloor \frac{n}{5} \rfloor))$ show $T(n) \leq c(n \log n)$
- $T(n) \leq 5T(\lfloor \frac{n}{5} \rfloor) + 6 = 5(c(\lfloor \frac{n}{5} \rfloor \log(\lfloor \frac{n}{5} \rfloor))) + 6 = cn \log(\frac{n}{5}) + 6$
- $T(n) \leq cn \log n - cn \log 5 + 6 \leq cn \log n - 3cn + 6$
- $T(n) \leq cn \log n$ if $c \geq 2$
- Base case:$T(1) \leq c \cdot 1$. True if $c \geq k$. Let $c = max(k, 2) n_0 = 1$