

# 6. Dynamic Programming

## Algorithmic Paradigms

**Greed.** Build up a solution incrementally, myopically optimizing some local criterion.

**Divide-and-conquer.** Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

**Dynamic programming.** Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

# Dynamic Programming Applications

## Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, systems, ....

## Some famous dynamic programming algorithms.

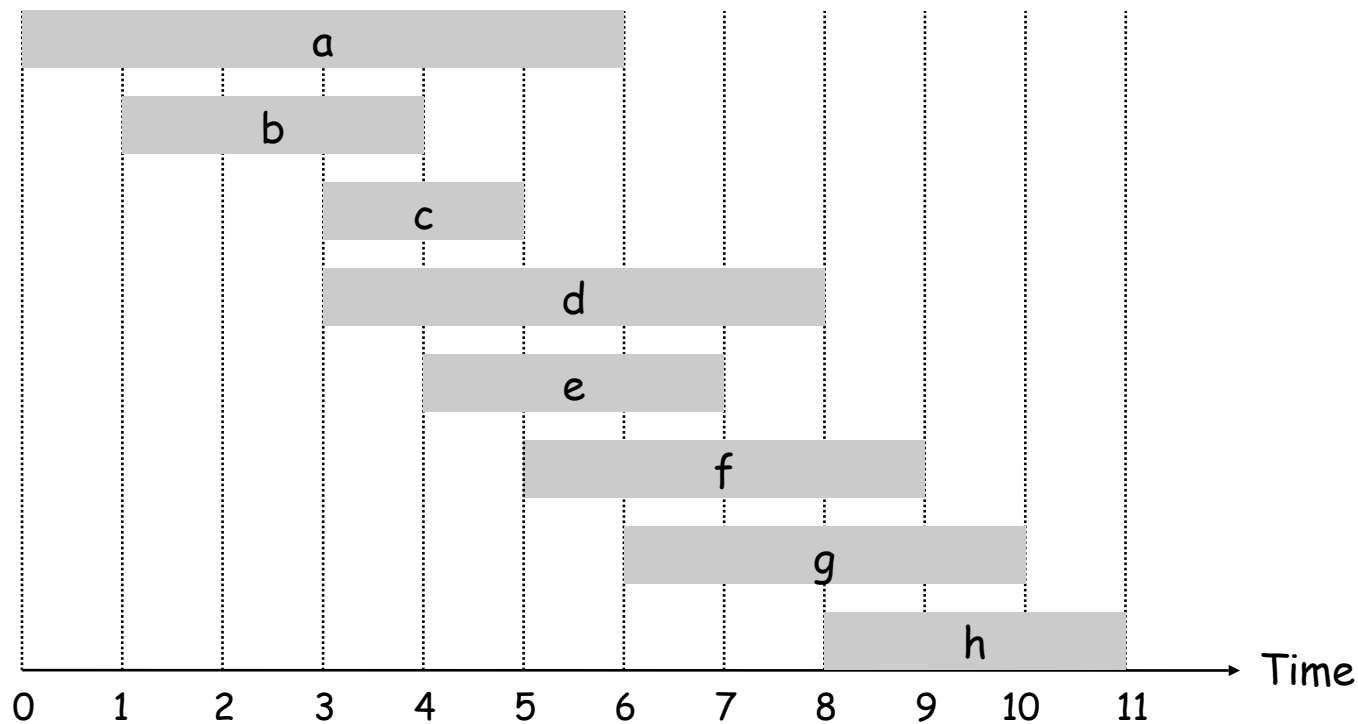
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

# 6.1 Weighted Interval Scheduling

## Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job  $j$  starts at  $s_j$ , finishes at  $f_j$ , and has weight or value  $v_j$ .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum **weight** subset of mutually compatible jobs.

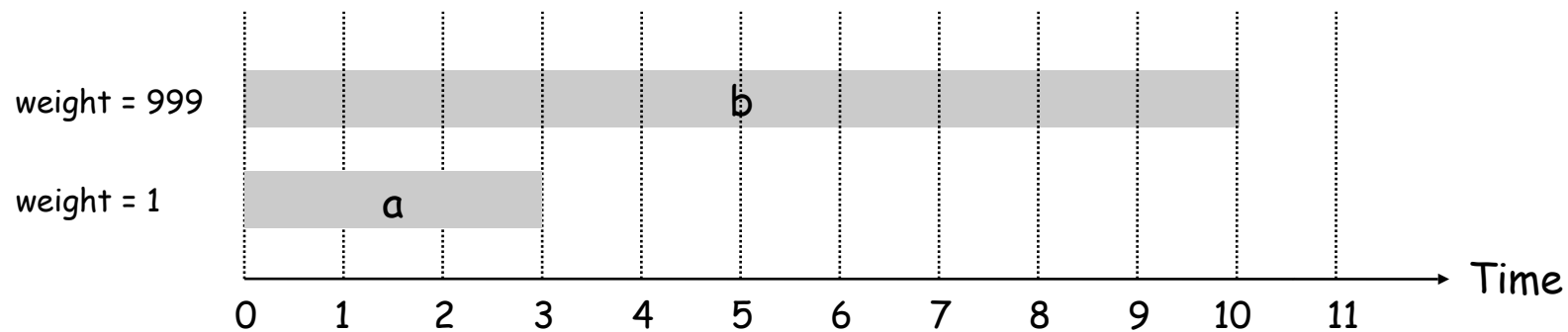


## Unweighted Interval Scheduling Review

**Recall.** Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

**Observation.** Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

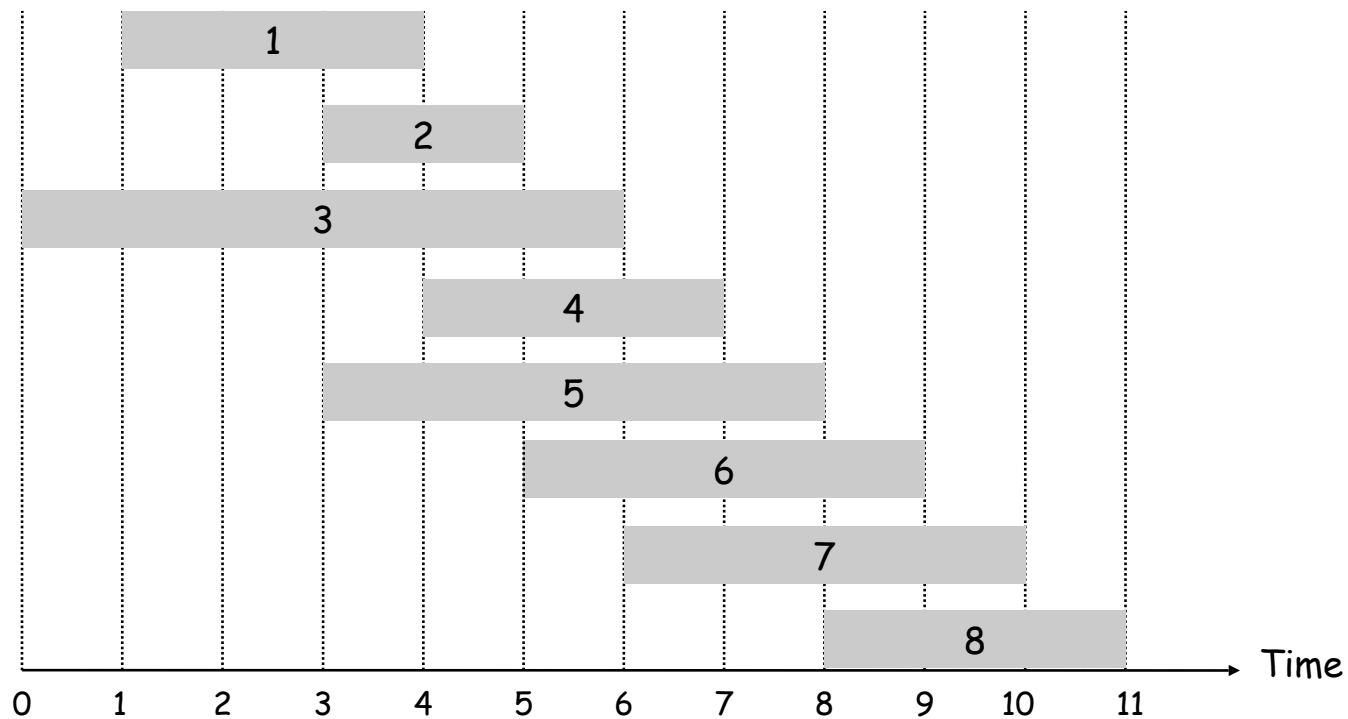


## Weighted Interval Scheduling

**Notation.** Label jobs by finishing time:  $f_1 \leq f_2 \leq \dots \leq f_n$ .

**Def.**  $p(j)$  = largest index  $i < j$  such that job  $i$  is compatible with  $j$ .

**Ex:**  $p(8) = 5$ ,  $p(7) = 3$ ,  $p(2) = 0$ .



## Dynamic Programming: Binary Choice

**Notation.**  $OPT(j)$  = value of optimal solution to the problem consisting of job requests  $1, 2, \dots, j$ .

- Case 1:  $OPT$  selects job  $j$ .
  - can't use incompatible jobs  $\{ p(j) + 1, p(j) + 2, \dots, j - 1 \}$
  - must include optimal solution to problem consisting of remaining compatible jobs  $1, 2, \dots, p(j)$
- Case 2:  $OPT$  does not select job  $j$ .
  - must include optimal solution to problem consisting of remaining compatible jobs  $1, 2, \dots, j-1$

↙  
↘  
optimal substructure

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)), OPT(j-1) \} & \text{otherwise} \end{cases}$$



## Weighted Interval Scheduling: Brute Force

Brute force algorithm.

```
Input:  $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$ 
```

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

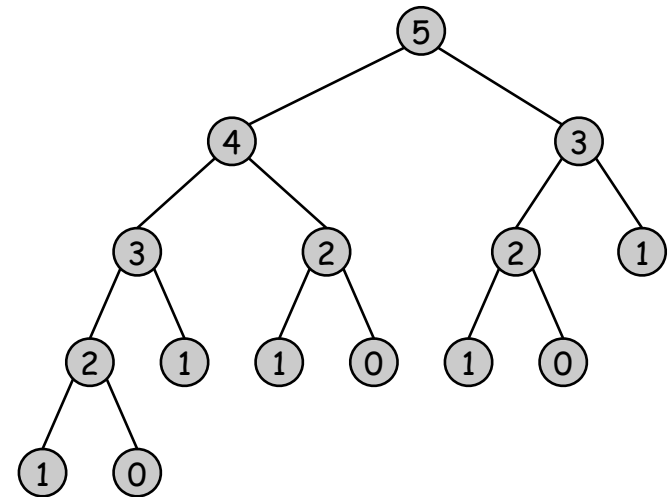
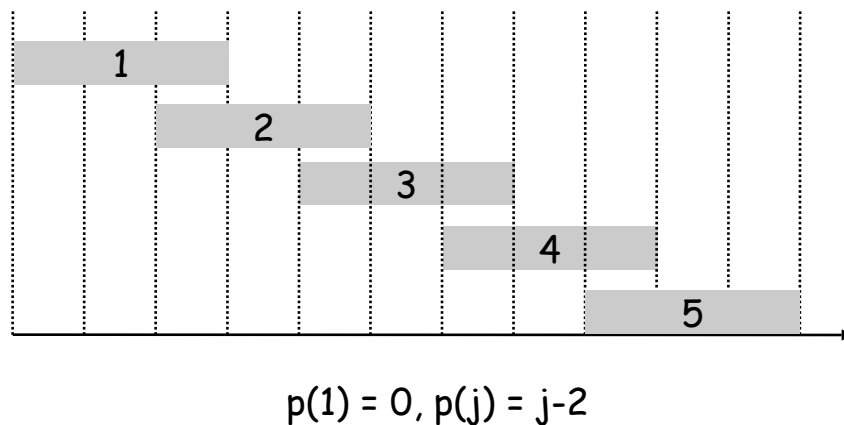
```
Compute  $p(1), p(2), \dots, p(n)$ 
```

```
Compute-Opt(j) {  
    if (j = 0)  
        return 0  
    else  
        return max( $v_j + \text{Compute-Opt}(p(j))$ ,  $\text{Compute-Opt}(j-1)$ )  
}
```

## Weighted Interval Scheduling: Brute Force

**Observation.** Recursive algorithm fails spectacularly because of redundant sub-problems  $\Rightarrow$  exponential algorithms.

**Ex.** Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



## Weighted Interval Scheduling: Memoization

**Memoization.** Store results of each sub-problem in a cache; lookup as needed.

```
Input:  $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$ 
```

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

```
Compute  $p(1), p(2), \dots, p(n)$ 
```

```
for  $j = 1$  to  $n$ 
```

```
     $M[j] = \text{empty}$  ← global array
```

```
 $M[j] = 0$ 
```

```
M-Compute-Opt( $j$ ) {
```

```
    if ( $M[j]$  is empty)
```

```
         $M[j] = \max(w_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$ 
```

```
    return  $M[j]$ 
```

```
}
```

## Weighted Interval Scheduling: Running Time

**Claim.** Memoized version of algorithm takes  $O(n \log n)$  time.

- Sort by finish time:  $O(n \log n)$ .
- Computing  $p(\cdot)$ :  $O(n)$  after sorting by start time.
- $M\text{-Compute-Opt}(j)$ : each invocation takes  $O(1)$  time and either
  - (i) returns an existing value  $M[j]$
  - (ii) fills in one new entry  $M[j]$  and makes two recursive calls
- Progress measure  $\Phi = \#$  nonempty entries of  $M[\ ]$ .
  - initially  $\Phi = 0$ , throughout  $\Phi \leq n$ .
  - (ii) increases  $\Phi$  by 1  $\Rightarrow$  at most  $2n$  recursive calls.
- Overall running time of  $M\text{-Compute-Opt}(n)$  is  $O(n)$ . ▪

**Remark.**  $O(n)$  if jobs are pre-sorted by start and finish times.

## Automated Memoization

**Automated memoization.** Many functional programming languages (e.g., Lisp) have library or built-in support for memoization.

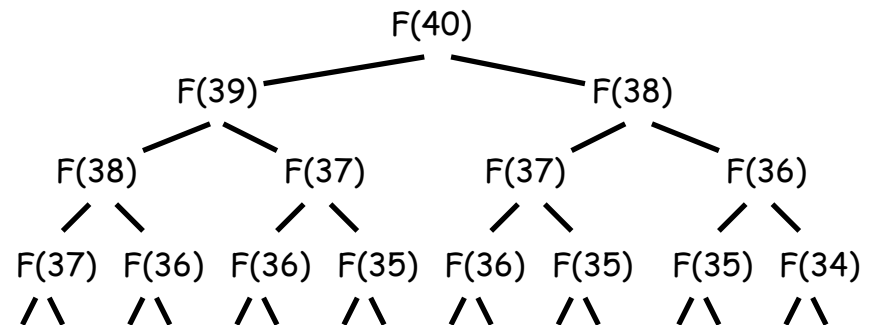
Q. Why not in imperative languages (e.g., Java)?

```
(def-memo-fun F (n)
  (if
    (<= n 1)
    n
    (+ (F (- n 1)) (F (- n 2)))))
```

Lisp (efficient)

```
static int F(int n) {
  if (n <= 1) return n;
  else return F(n-1) + F(n-2);
}
```

Java (exponential)



## Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
  if (j = 0)
    output nothing
  else if ( $v_j + M[p(j)] > M[j-1]$ )
    print j
    Find-Solution(p(j))
  else
    Find-Solution(j-1)
}
```

- # of recursive calls  $\leq n \Rightarrow O(n)$ .

## Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

```
Input:  $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$ 
```

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

```
Compute  $p(1), p(2), \dots, p(n)$ 
```

```
Iterative-Compute-Opt {  
     $M[0] = 0$   
    for  $j = 1$  to  $n$   
         $M[j] = \max(v_j + M[p(j)], M[j-1])$   
}
```

## 6.3 Segmented Least Squares

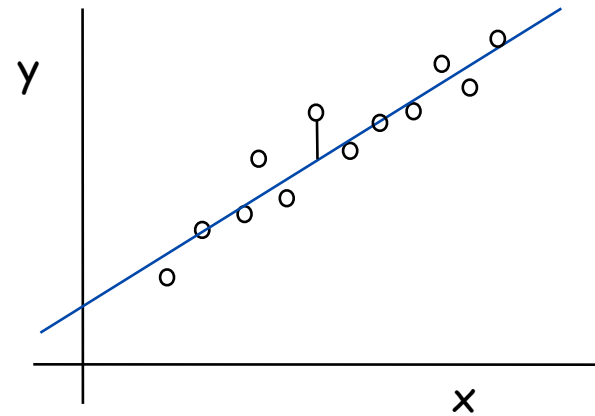


## Segmented Least Squares

### Least squares.

- Foundational problem in statistic and numerical analysis.
- Given  $n$  points in the plane:  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ .
- Find a line  $y = ax + b$  that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^n (y_i - ax_i - b)^2$$



**Solution.** Calculus  $\Rightarrow$  min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i) (\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

## Segmented Least Squares

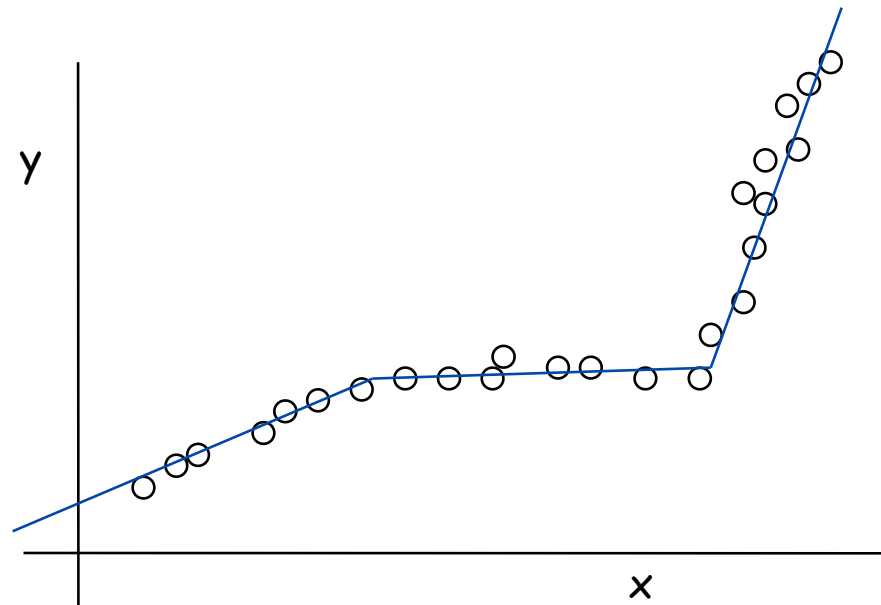
### Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given  $n$  points in the plane  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  with
- $x_1 < x_2 < \dots < x_n$ , find a sequence of lines that minimizes  $f(x)$ .

Q. What's a reasonable choice for  $f(x)$  to balance accuracy and parsimony?

↑  
number of lines

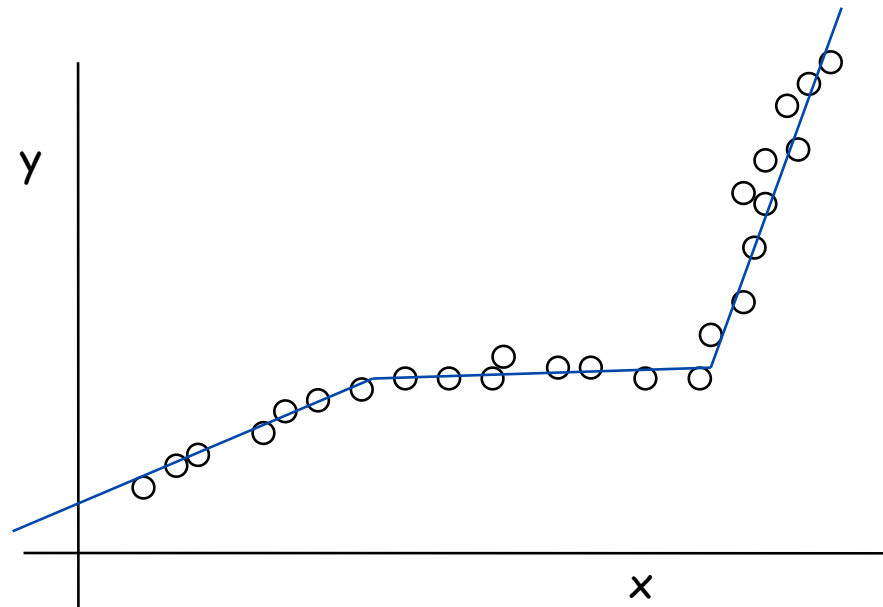
↑  
goodness of fit



## Segmented Least Squares

### Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given  $n$  points in the plane  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  with
- $x_1 < x_2 < \dots < x_n$ , find a sequence of lines that minimizes:
  - the sum of the sums of the squared errors  $E$  in each segment
  - the number of lines  $L$
- Tradeoff function:  $E + c L$ , for some constant  $c > 0$ .



## Dynamic Programming: Multiway Choice

### Notation.

- $OPT(j)$  = minimum cost for points  $p_1, p_{i+1}, \dots, p_j$ .
- $e(i, j)$  = minimum sum of squares for points  $p_i, p_{i+1}, \dots, p_j$ .

### To compute $OPT(j)$ :

- Last segment uses points  $p_i, p_{i+1}, \dots, p_j$  for some  $i$ .
- Cost =  $e(i, j) + c + OPT(i-1)$ .

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \leq i \leq j} \{ e(i, j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

## Segmented Least Squares: Algorithm

**INPUT:**  $n, p_1, \dots, p_N, c$

```
Segmented-Least-Squares() {  
  M[0] = 0  
  for j = 1 to n  
    for i = 1 to j  
      compute the least square error  $e_{ij}$  for  
      the segment  $p_i, \dots, p_j$   
  
  for j = 1 to n  
    M[j] =  $\min_{1 \leq i \leq j} (e_{ij} + c + M[i-1])$   
  
  return M[n]  
}
```

**Running time.**  $O(n^3)$ .  $\leftarrow$  can be improved to  $O(n^2)$  by pre-computing various statistics

- Bottleneck = computing  $e(i, j)$  for  $O(n^2)$  pairs,  $O(n)$  per pair using previous formula.