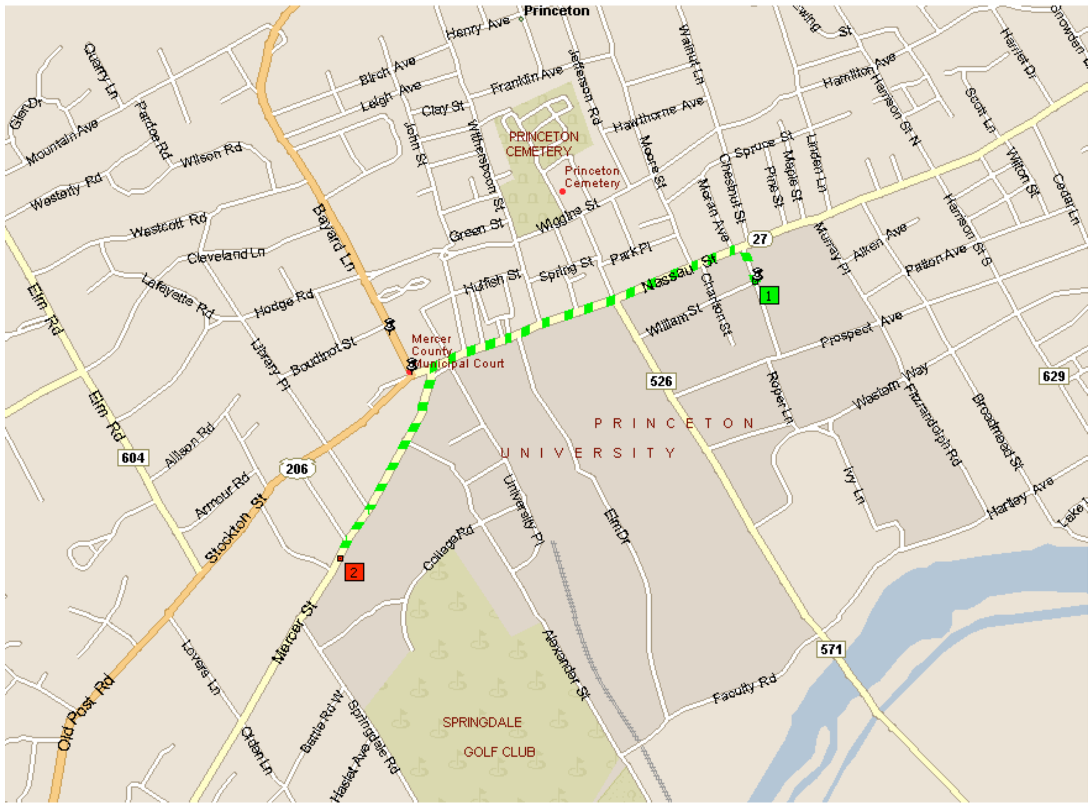
4.4 Shortest Paths in a Graph **Revisited**



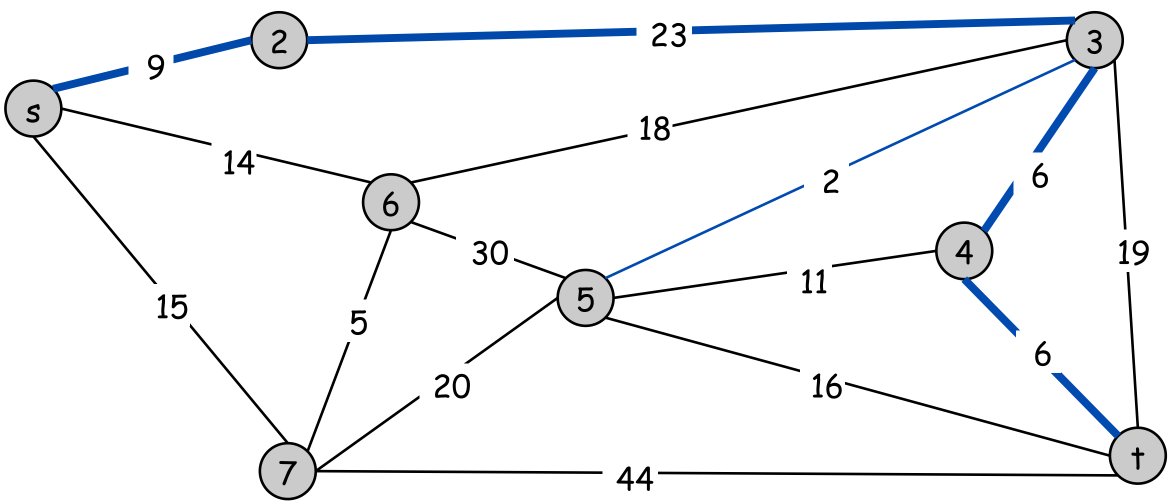
shortest path from computer science department to Einstein's house

Shortest Path Problem

Shortest path network.
- Directed graph $G = (V, E)$.
- Source s, destination t.
- Length ℓ_e = length of edge e ($\ell_e \geq 0$).

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 6 + 6
= 44.

Dijkstra's Algorithm

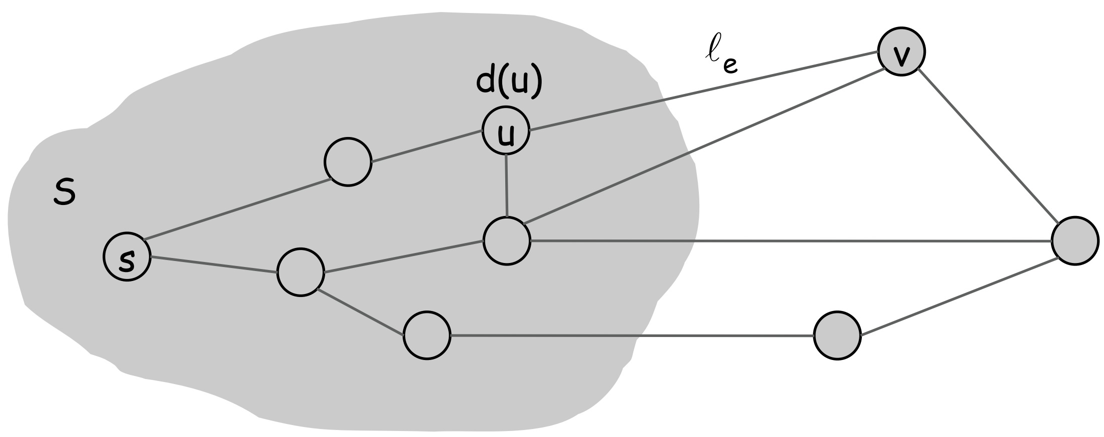
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = π(v).



Dijkstra's Algorithm

Dijkstra's algorithm.

- ```
  M = {}
  forall v ∈ V
    d(v) = π(v) = ∞
  π(s) = 0
  loop
  ```
  LI$_1$: forall $v \in M$, d(v) is the minimal distance from s to v and uses edges only in M
  ```
   exit when V=M
   Choose v from V-M with minimal π(v)
   Add v to M
   d(v) = π(v)
  end loop
  ```
  Postcondition: forall $v \in V$, d(v) is the minimal distance from s to v
  ```

Dijkstra's Algorithm—Proof of Correctness

Loop invariant initialization

Given:

```
M = {}
 forall v ∈ V
    d(v) = π(v) = ∞
 π(s) = 0
```

Show:

LI_1: forall v ∈ M, d(v) is the minimal distance from s to v

Since M is empty, LI1 is trivially true

Dijkstra's Algorithm—Proof of Correctness

Loop invariant maintentance

Given:

LI_1': forall $v \in M'$, $d'(v)$ is the minimal distance from s to v

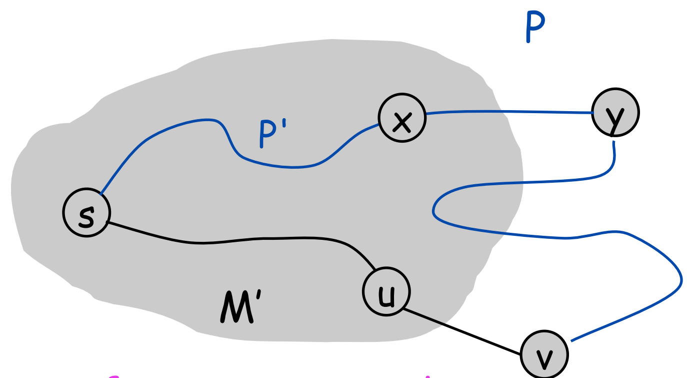
NOT Exit condition: `V≠M`

code

Show:

LI_1'': forall $v \in M''$, $d''(v)$ is the minimal distance from s to v

Dijkstra's Algorithm—Proof of Correctness



Loop invariant maintentance

Given:

LI_1', V≠M, LI_2', code

Show:

LI_1'': forall v ∈ M'', d''(v) is the minimal distance from s to v and uses edges only between verticies in M

M'' is M' plus one new node, v. The values of d for elements of M' are unchanged.

```
Choose v from V-M with minimal π(v)
 Add v to M
 d(v) = π(v)
```

For that one new node in M'', v, d(v) is set to π(v), which is the minimal distance from s to v, using nodes only in M' plus one extra edge. Any other path to v would need to go through some other node in V-M, y, which would have higher π. Since edge lengths are non-negative l(s-y-v) ≥ l(s-y) ≥ l(s-v). Therefore, π(v) is the minimal distance from s to v. The code sets d(v) to that minimal distance. In addition. d(u) uses edges only between vertices in M'' and v is in M''..

Dijkstra's Algorithm—Proof of Correctness

Loop termination

Given:

```
loop
 exit when V=M
 Choose v from V-M with minimal π(v)
 Add v to M
 d(v) = π(v)
end loop
```

Show:

exit condition (M=V) is eventually satisfied

Every time through the loop |M| increases by one

Dijkstra's Algorithm—Proof of Correctness

Postcondition correctness

Given:

LI_1: forall $v \in M$, $d(v)$ is the minimal distance from s to v

Exit condition: `V=M`

Show:

Postcondition: forall $v \in V$, $d(v)$ is the minimal distance from s to v

Since the exit condition shows that $V=M$, we can rewrite replace M with V in LI_1, yielding the postcondition

Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{ \ \pi(w), \ \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

PQ Operation	Dijkstra	Binary heap
Insert	n	log n
ExtractMin	n	log n
ChangeKey	m	log n
IsEmpty	n	1
Total		m log n

Dijkstra's Algorithm—Runtime analysis

Executes the for-loop n times

Each time through the loop, must:

- remove from set—O(1)
- find minimum π(v)
- update π(v) for all edges adjoining v.

If we keep π(v) in a priority queue, we can do the following

- Before loop Insert all vertices in priority queue—O(n log n)
- During loop (executed n times)
  - Check Priority Queue IsEmpty—O(1)*n = O(n)
  - ExtractMinimum π(v)—O(log n)*n = O(n log n)
  - For each edge adjoining v, ChangeKey—O(log n)
    - ✐ How many times do we execute?
    - ✐ We will use each directed edge only once in the algorithm
    - ✐ O(m log n) (total, not per loop execution)
- T(n) = max(O(n log n), O(n), O(n log n), O(m log n)) = O(m log n)

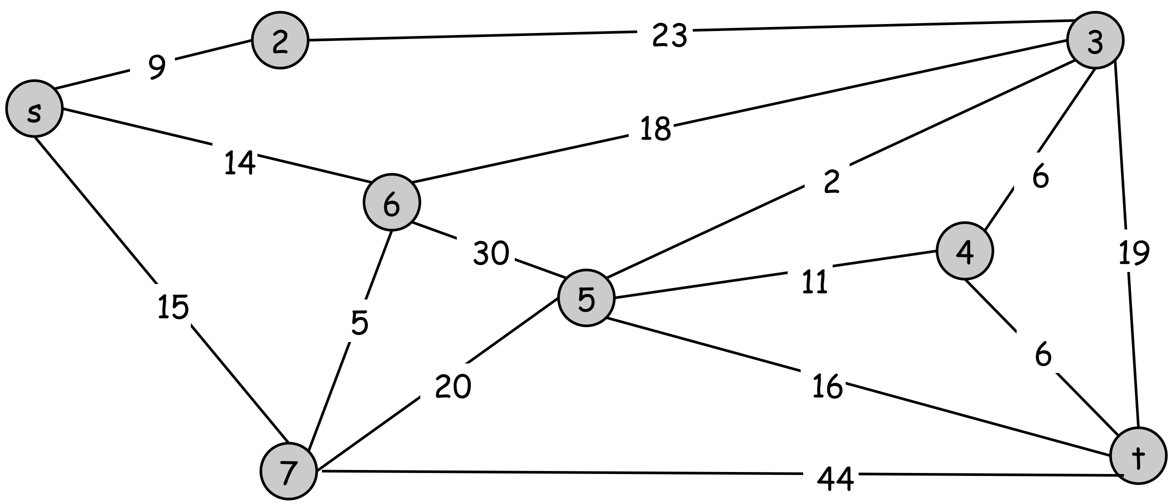
Given minimal distances, how to find minimal path?

Work backwards

Find adjacent vertex such that $d(u) + l(u, v) = d(v)$

$d(t) = 44 = d(4) + 6$
$d(4) = 38 = d(3) + 6$
$d(3) = 32 = d(2) + 23$
$d(2) = 9 = d(s) + 9$
done!

For each vertex, find adjoining edges
$T(n, m) = O(n + m) = O(m)$



Steps for presenting an algorithm

Provide algorithm pseudo-code

All these parts are needed for homework or tests!

Prove correctness

- Loop invariant
  - Initialization
    - Follows from precondition and pre-loop code
  - Maintenance
    - Follows from loop invariant and loop code and not(exit condition)
  - Loop Termination
    - Define some measure of progress
    - Verify progress made in loop
- Postcondition correctness
  - Follows from Loop invariant, exit condition, and post-loop code

Runtime Analysis

- Provide T(n)=O(...)
- Make bound as tight as possible