4. Greed

> Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.
>
> \- Gordon Gecko (Michael Douglas)

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: $.89

Coin-Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: $c_1 < c_2 < … < c_n$.

   coins selected

S ← $\phi$
while ($x \neq 0$) {
   let k be largest integer such that $c_k \leq x$
   if (k = 0)
      return "no solution found"
   $x \leftarrow x - c_k$
   S ← S ∪ {k}
}
return S
```

Q. Is cashier's algorithm optimal?

Coin-Changing: Analysis of Greedy Algorithm

Theorem. Greed is optimal for U.S. coinage: 1, 5, 10, 25

Pf.

- Optimum has ≤4 pennies (will choose 5 instead of 5*1)
- Optimum has ≤1 nickel (will choose 1*10 instead of 2*5)
- Optimum has ≤ 2 dimes (Optimum will choose 25+5 instead of 3*10)
- Optimum doesn't have 2 dimes and a nickel (otherwise, will choose 25)
- Opt & greedy have at most 24 in P, N, D. Above 24 uses as many Q as possible: remainder P, N, D
- By inspection, Opt & Greedy agree on 1..24

k	c_k	All optimal solutions must satisfy	Max value of coins 1, 2, …, k-1 in any OPT
1	1	P ≤ 4	-
2	5	N ≤ 1	4
3	10	N + D ≤ 2	4 + 5 = 9
4	25	Q ≤ 3	20 + 4 = 24

Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal without nickels.

Counterexample. 30¢.

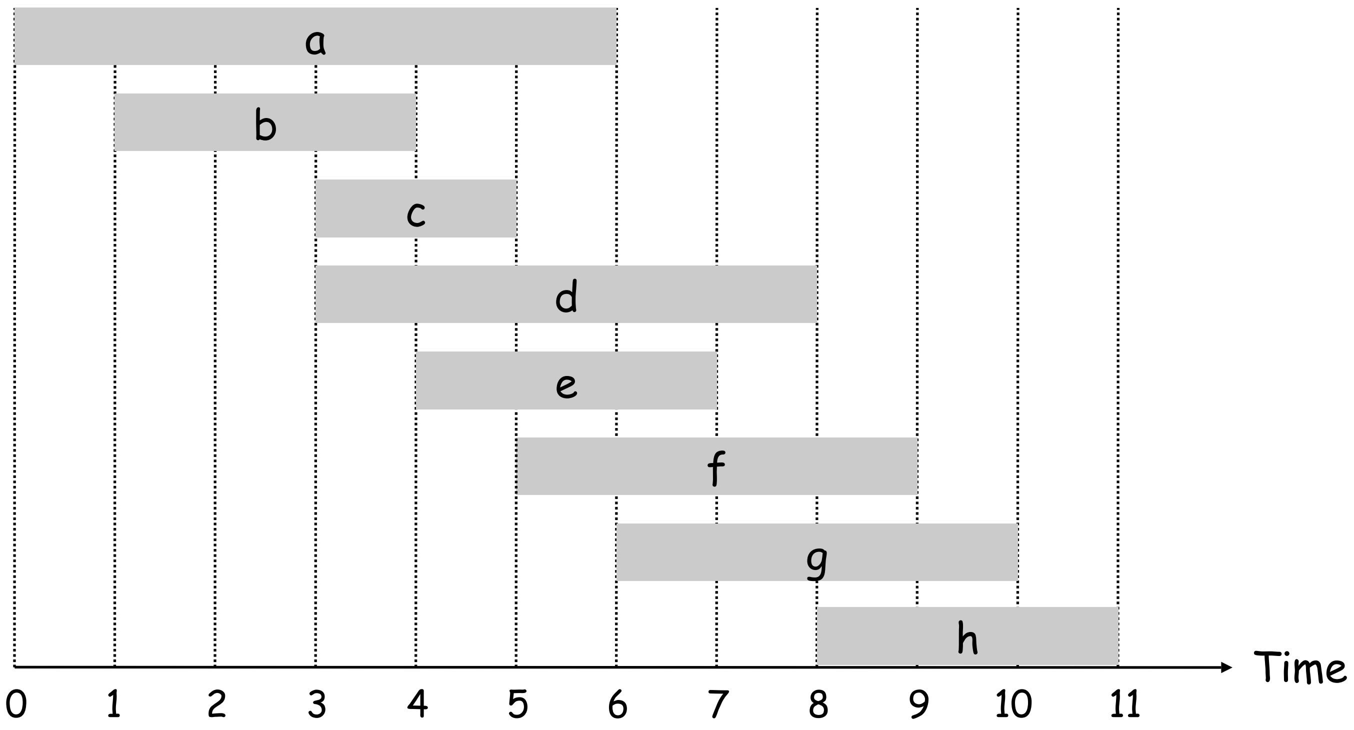
- Greedy: 25, 1, 1, 1, 1, 1.
- Optimal: 10, 10, 10.

4.1 Interval Scheduling

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time s_j.
- [Earliest finish time] Consider jobs in ascending order of finish time f_j.
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

breaks earliest start time

breaks shortest interval

breaks fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

  jobs selected

A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

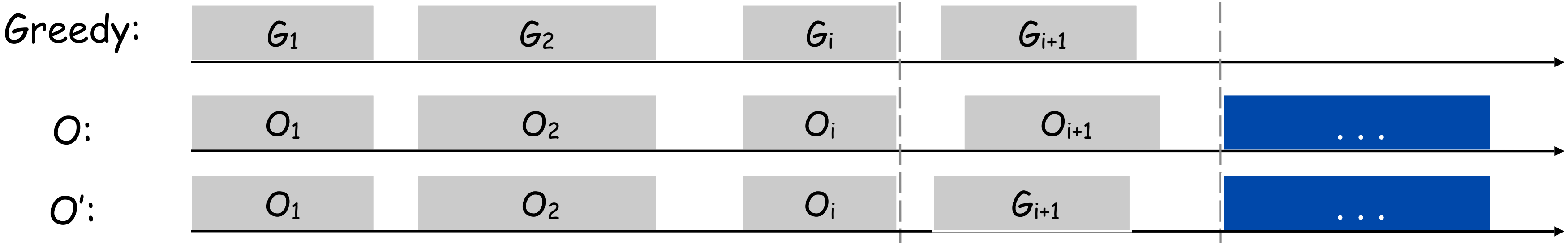
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by induction)

- There is an optimal solution O whose choices $O_1..O_n$ match the greedy algorithm's choices $G_1..G_n$
- Base case (first job)
  - Take any optimal solution O. If $O_1 \neq G_1$, then replace G_1 with O_1 (it remains compatible since $start(O_2) > finish(O_1) \geq finish(G_1)$
- Induction (assume there is an optimal solution O whose choices $O_1..O_i$ match $G_1..G_i$). Show there exists an optimal solution O' starting with $G_1..G_{i+1}$
  - If $O_{i+1} <> G_{i+1}$, then $finish(O_{i+1}) \geq finish(G_{i+1})$ (by algorithm)
  - $O_1..O_i, G_{i+1}, O_{i+2}, \ldots O_n$ is compatible and optimal (splice-in)
- Greedy algorithm can't have fewer choices than optimal (by algorithm)



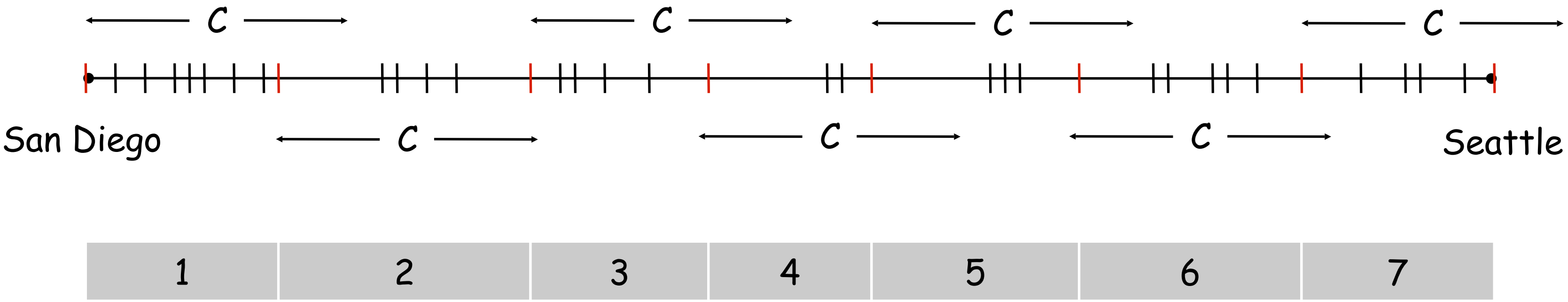
Selecting Breakpoints

Selecting Breakpoints

Selecting breakpoints.

- Road trip from San Diego to Seattle along fixed route.
- Refueling stations at certain points along the way.
- Fuel capacity = C.
- Goal: makes as few refueling stops as possible.

Greedy algorithm. Go as far as you can before refueling.



Selecting Breakpoints: Greedy Algorithm

Truck driver's algorithm.

```
Sort breakpoints so that: $0 = b_0 < b_1 < b_2 < ... < b_n = L$

S ← {0}   ⟵  breakpoints selected
x ← 0     ⟵  current location

while (x < $b_n$)
   let p be largest integer such that $b_p \le x + C$
   if ($b_p$ = x)
      return "no solution"
   x ← $b_p$
   S ← S ∪ {p}
return S
```

Implementation. O(n log n)

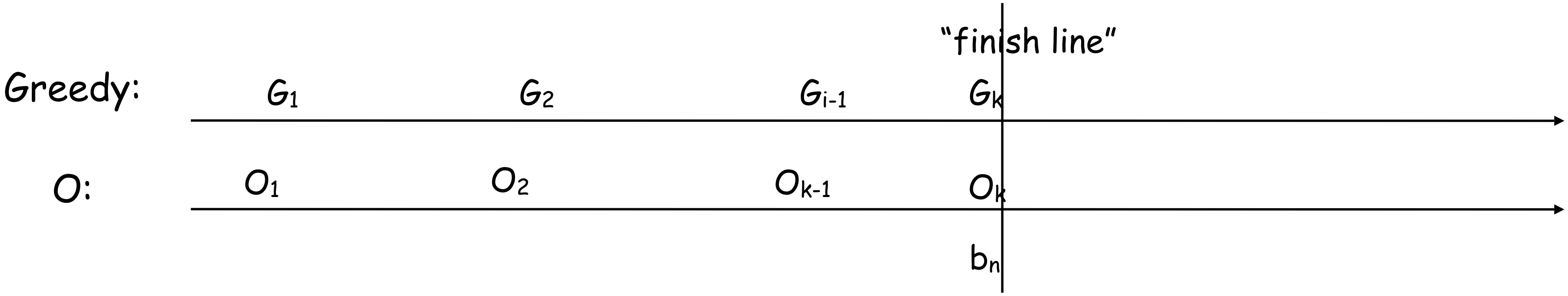
- Use binary search to select each breakpoint p.

Selecting Breakpoints: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by induction)

- Lemma: For any optimal solution, $O_i <= G_i$, for $1 \leq i \leq k$ (k = # breakpoints in O) (proof by induction)
- Base case (first job)
  - G_1 is maximum possible, so $O_1 <= G_1$
- Induction (assume that for any optimal solution O, $O_{i-1} \leq G_{i-1}$). Show $O_i \leq G_i$
  - If $O_{i+1} > G_{i+1}$, then greedy algorithm would have chosen it. (algorithm chooses largest possible)
- $|G|=|O|$. If not, then there's a G_{k+1}. But $O_k=b_n$ and $G_k >= O_k$ (by lemma). But, algorithm stops when $G_i >= b_n$, so there's no such G_{k+1}.



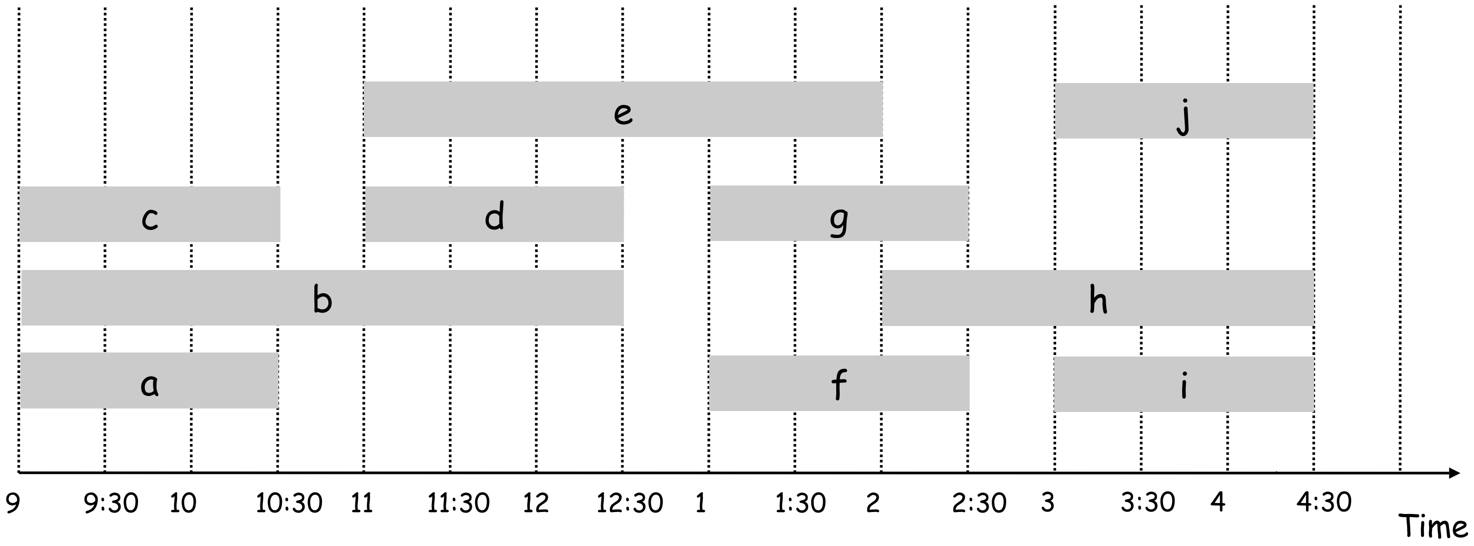
4.1 Interval Partitioning

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

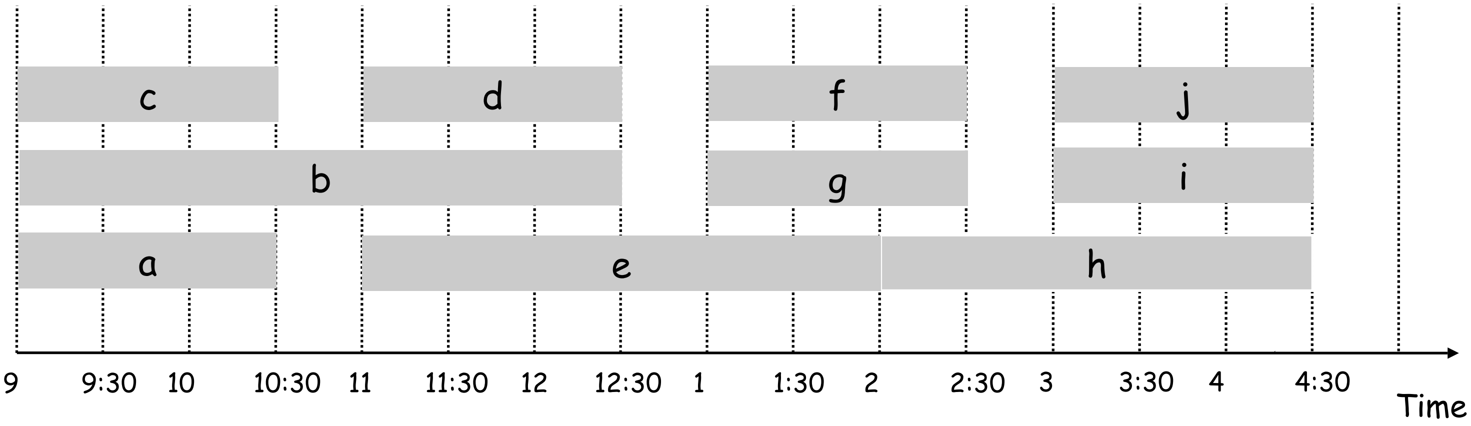


Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

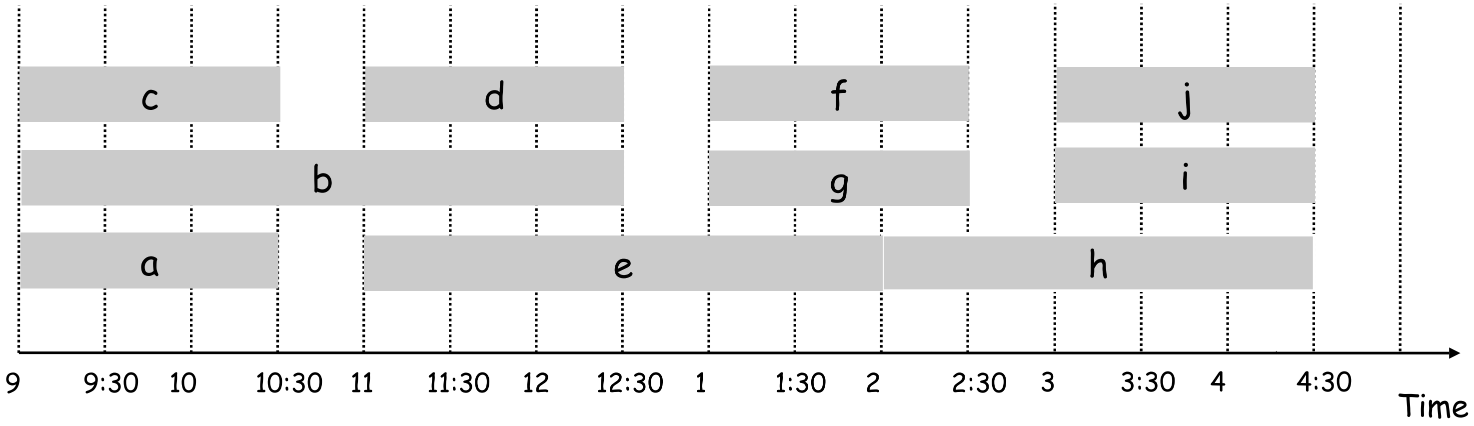
Def. The **depth** of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed ≥ depth.

Ex: Depth of schedule below = $\underset{\uparrow}{3}$ ⇒ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that $s_1 \le s_2 \le ... \le s_n$.
d ← 0    ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation ⇒ all schedules use ≥ d classrooms. ▪

4.2 Scheduling to Minimize Lateness

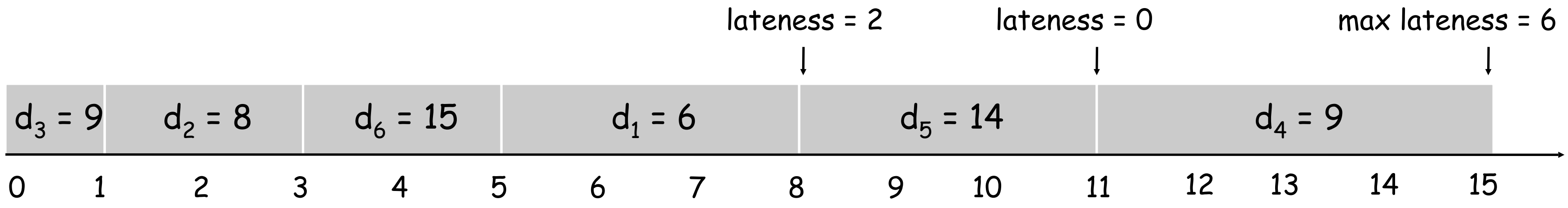
Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, \; f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

- [Earliest deadline first] Consider jobs in ascending order of deadline d_j.

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

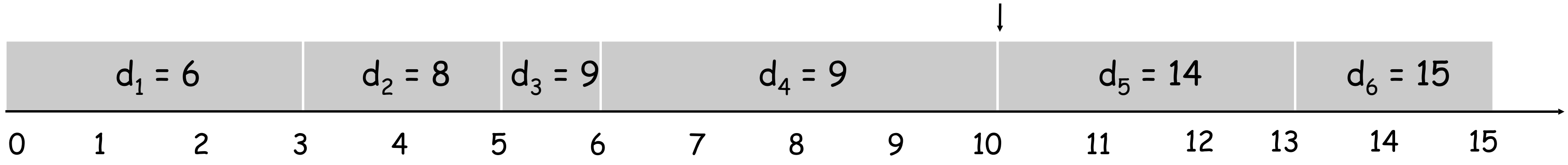
counterexample

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

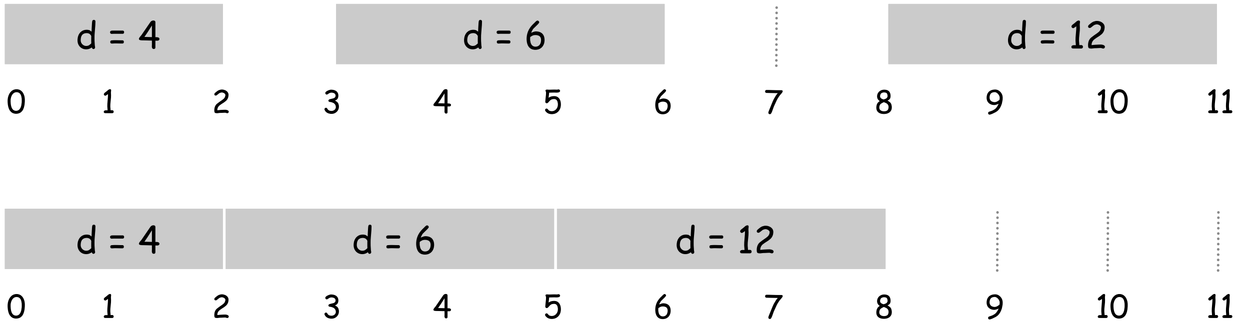
```
Sort n jobs by deadline so that $d_1 \le d_2 \le \ldots \le d_n$


t ← 0
for j = 1 to n
   Assign job j to interval [t, t + $t_j$]
   $s_j$ ← t, $f_j$ ← t + $t_j$
   t ← t + $t_j$
output intervals [$s_j$, $f_j$]
```

max lateness = 1



Minimizing Lateness: No Idle Time

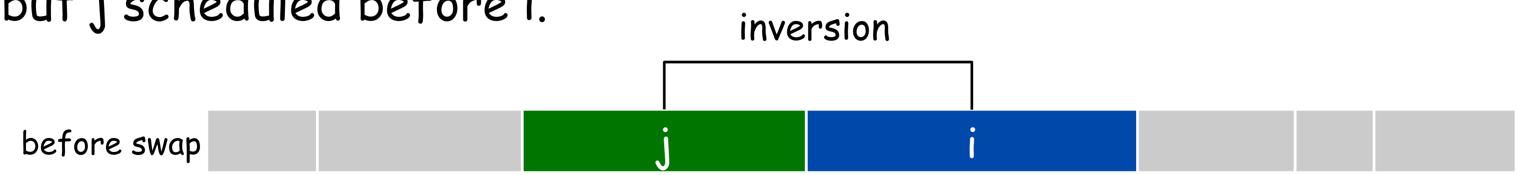
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.

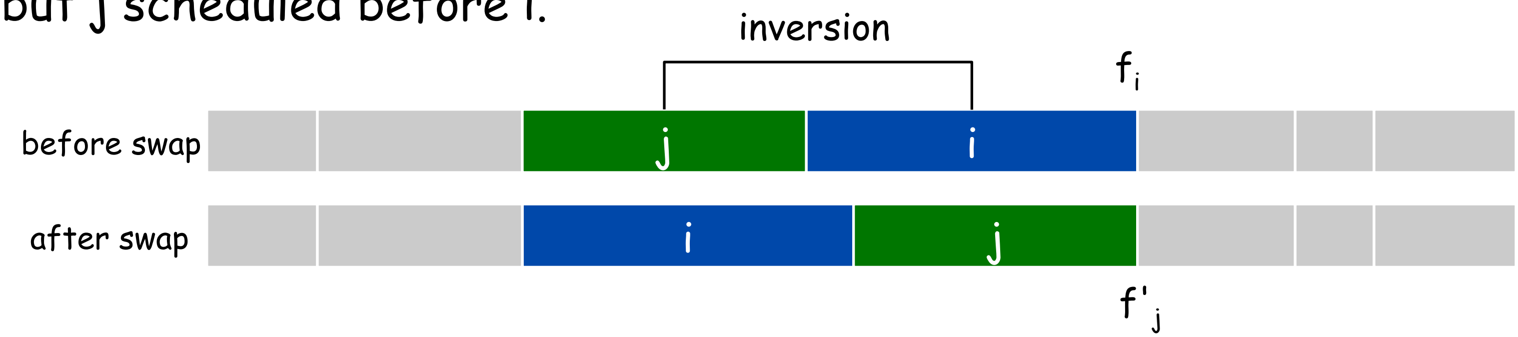


Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.



Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && \text{(j finishes at time } f_i\text{)} \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*.
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

4.3 Optimal Caching

Optimal Offline Caching

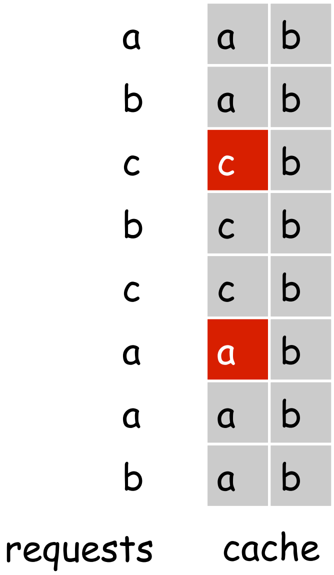
Caching.

- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Ex: k = 2, initial cache = ab,
requests: a, b, c, b, c, a, a, b.

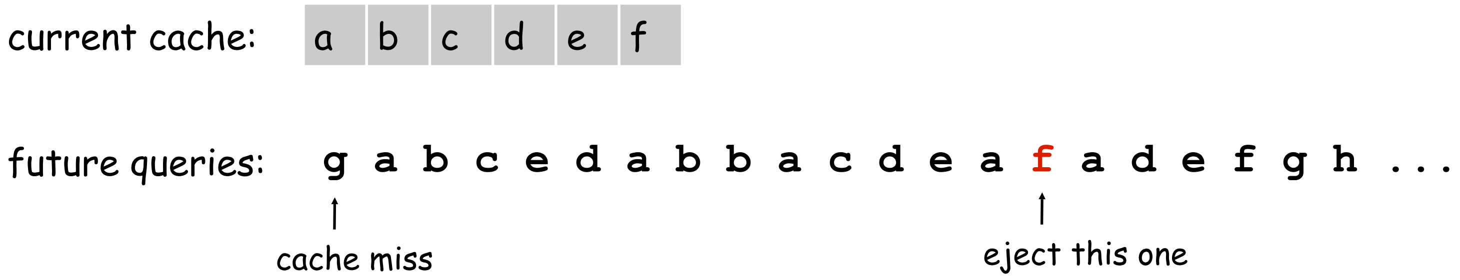
Optimal eviction schedule: 2 cache misses.



a	a	b
b	a	b
c	c	b
b	c	b
c	c	b
a	a	b
a	a	b
b	a	b
requests	cache	

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.



current cache: a b c d e f

future queries: `g a b c e d a b b a c d e a f a d e f g h ...`

cache miss

eject this one

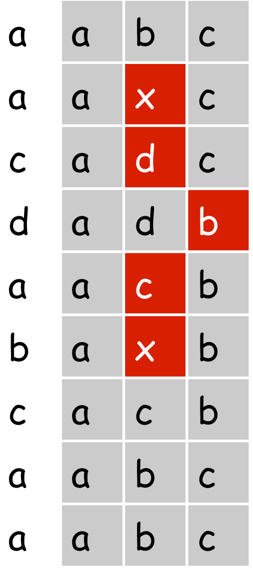
Theorem. [Bellady, 1960s] FF is optimal eviction schedule.

Pf. Algorithm and theorem are intuitive; proof is subtle.

Reduced Eviction Schedules

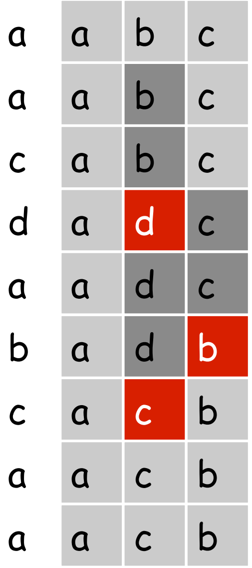
Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.



a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

an unreduced schedule



a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

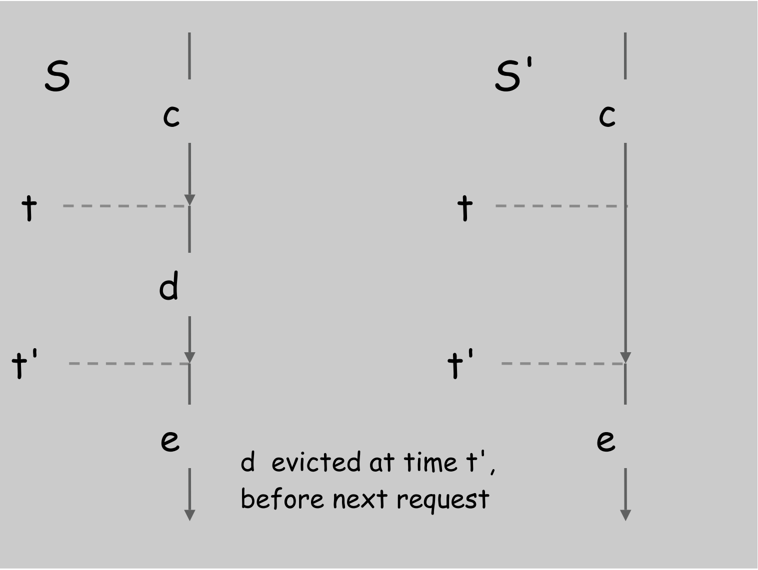
Reduced Eviction Schedules

Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more cache misses.

doesn't enter cache at requested time

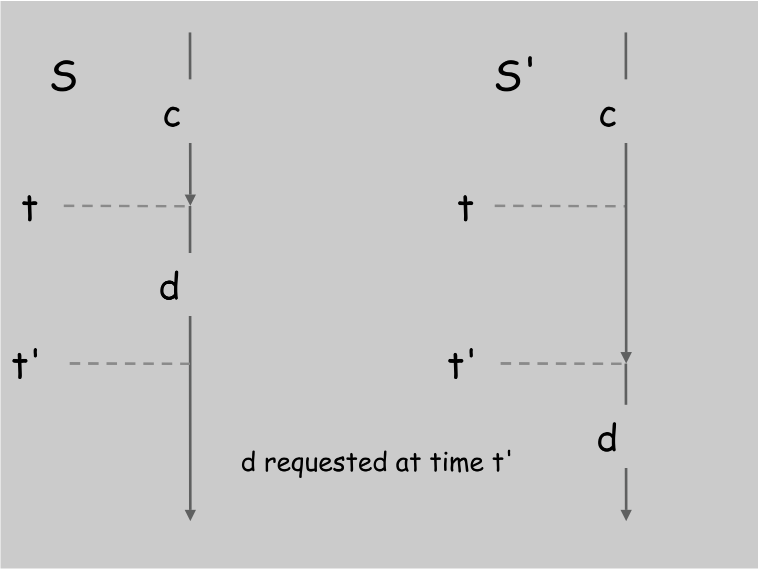
Pf. (by induction on number of unreduced items)

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- Case 1: d evicted at time t', before next request for d.
- Case 2: d requested at time t' before d is evicted. ▪



d evicted at time t', before next request

Case 1



d requested at time t'

Case 2

Farthest-In-Future: *Analysis*

Theorem. FF is optimal eviction algorithm.
Pf. (by induction on number or requests j)

> Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first j+1 requests.

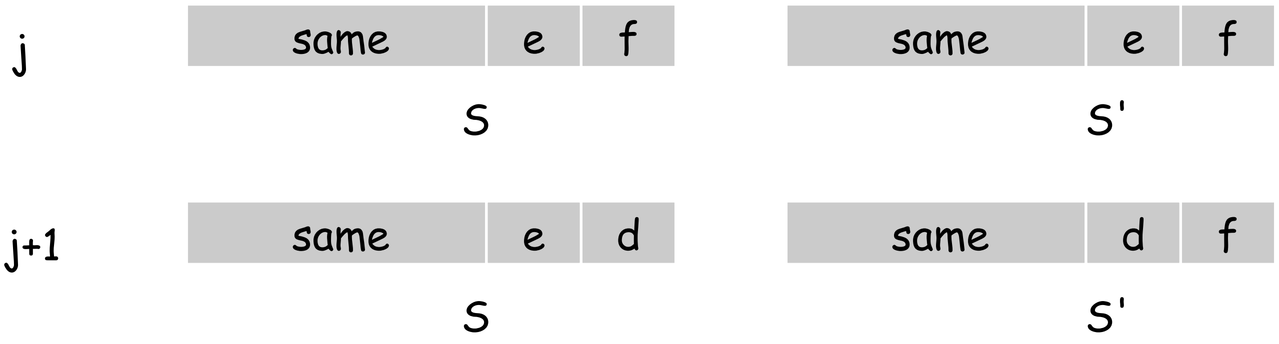
Let S be reduced schedule that satisfies invariant through j requests.
We produce S' that satisfies invariant after j+1 requests.

- Consider $(j+1)^{st}$ request $d = d_{j+1}$.
- Since S and S_{FF} have agreed up until now, they have the same cache contents before request j+1.
- Case 1: (d is already in the cache). S' = S satisfies invariant.
- Case 2: (d is not in the cache and S and S_{FF} evict the same element).
  S' = S satisfies invariant.

Farthest-In-Future: *Analysis*

Pf. (continued)

- Case 3: (d is not in the cache; S_{FF} evicts e; S evicts f ≠ e).
  - begin construction of S' from S by evicting e instead of f

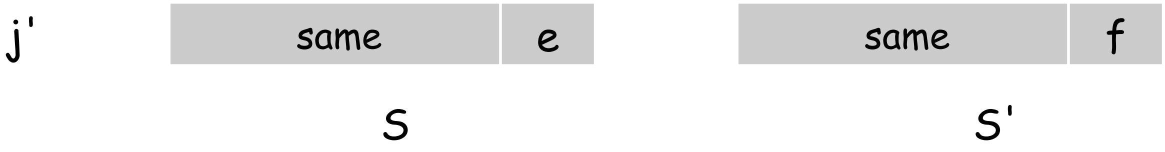


  - now S' agrees with S_{FF} on first j+1 requests; we show that having element f in cache is no worse than having element e

Farthest-In-Future: *Analysis*

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



- Case 3a: g = e. Can't happen with Farthest-In-Future since there must be a request for f before e.

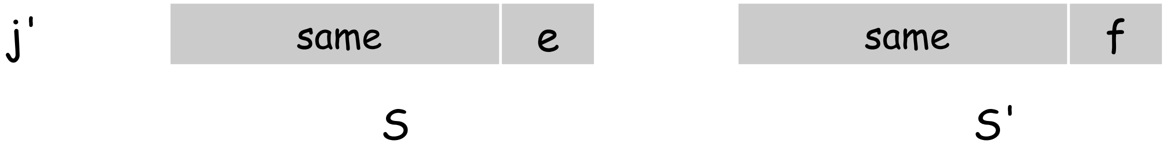
- Case 3b: g = f. Element f can't be in cache of S, so let e' be the element that S evicts.
  - if e' = e, S' accesses f from cache; now S and S' have same cache
  - if e' ≠ e, S' evicts e' and brings e into the cache; now S and S' have the same cache

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with S_{FF} through step j+1

Farthest-In-Future: *Analysis*

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



otherwise S' would take the same action

- Case 3c: g ≠ e, f. S must evict e. Make S' evict f; now S and S' have the same cache. ▪

Caching Perspective

Online vs. offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

LRU. Evict page whose most recent access was earliest.

↑

FF with direction of time reversed!

Theorem. FF is optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.