Notes on Corner Detection

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1 Formulating the Cost Function

Consider a neighborhood of an image containing an ideal corner. As drawn in Figure 1(a), the tangent lines to the shape intersect exactly at the location of the corner.

In real images, the corners aren't this clean and we need an approximate solution. We will seek a least-squares solution for the point of intersection of the tangent lines, as suggested in Figure 1(b).

The equation for the tangent line $\ell_{\mathbf{x}'}$ passing through a pixel at location $\mathbf{x}'$ is

$$D_{\mathbf{x}'}(\mathbf{x}) = \nabla I(\mathbf{x}')^\top (\mathbf{x} - \mathbf{x}') = 0$$

where $\nabla I(\mathbf{x}')$ denotes the gradient of the image I at $\mathbf{x}'$, indicated by the small arrow in Figure 1(b). Our goal is to find the point $\mathbf{x}_o$ with the minimum perpendicular distance to all the lines in this neighborhood:

$$\mathbf{x}_o = \arg\min_{\mathbf{x}\in\mathbb{R}^2} \int_{\mathbf{x}'\in\mathcal{N}} D_{\mathbf{x}'}(\mathbf{x})^2 d\mathbf{x}'$$

where $\mathcal{N}$ denotes the neighborhood around a candidate corner location. This expression is the weighted integral of the squares of the distances from $\mathbf{x}$ to all lines in $\mathcal{N}$. $D_{\mathbf{x}'}(\mathbf{x})$

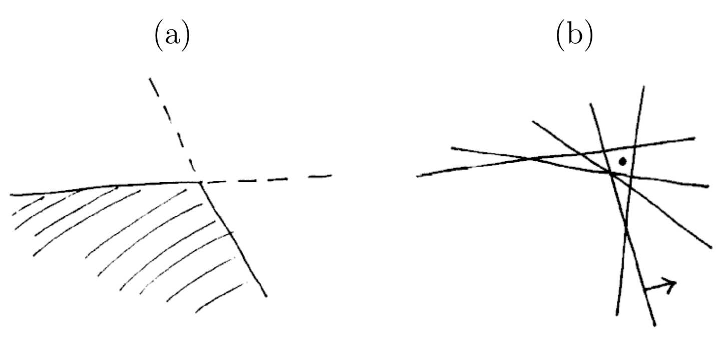

Figure 1: (a) Ideal case: tangent lines intersect exactly at corner. (b) In real images, we can only hope to find an approximation to this point of intersection, indicated here by the dot. The arrow indicates a gradient vector at a sample pixel.

describes the distance from $\mathbf{x}$ to the line $\ell_{\mathbf{x}'}$ multiplied by the gradient magnitude. In this way we give more weight to lines that pass through strong edge pixels. (Recall that pixels not on edges tend to have arbitrary gradient directions but small gradient magnitudes.)

2 The least-squares solution

We can find $\mathbf{x}_o$ in closed form as follows:

$$\begin{aligned} \mathbf{x}_o &= \arg\min_{\mathbf{x}\in\mathbb{R}^2} \int \left(\nabla I(\mathbf{x}')^\top(\mathbf{x}-\mathbf{x}')\right)^2 d\mathbf{x}' \\ &= \arg\min_{\mathbf{x}} \int (\mathbf{x}-\mathbf{x}')^\top \nabla I(\mathbf{x}')\nabla I(\mathbf{x}')^\top(\mathbf{x}-\mathbf{x}')d\mathbf{x}' \\ &= \arg\min_{\mathbf{x}} \mathbf{x}^\top A\mathbf{x} - 2\mathbf{x}^\top\mathbf{b} + c \end{aligned} \tag{1}$$

where

$$\begin{aligned} A &= \int \nabla I(\mathbf{x}')\nabla I(\mathbf{x}')^\top d\mathbf{x}' \\ \mathbf{b} &= \int \nabla I(\mathbf{x}')\nabla I(\mathbf{x}')^\top \mathbf{x}' d\mathbf{x}' \\ c &= \int \mathbf{x}'^\top \nabla I(\mathbf{x}')\nabla I(\mathbf{x}')^\top \mathbf{x}' d\mathbf{x}' \end{aligned}$$

The arg min in Equation (1) can be found by taking its derivative w.r.t. $\mathbf{x}$ and setting it equal to zero. This gives

$$2A\mathbf{x} - 2\mathbf{b} = \mathbf{0} \quad \Longrightarrow \quad A\mathbf{x} = \mathbf{b}$$

at the minimum. Therefore $\mathbf{x}_o = A^{-1}\mathbf{b}$.

Of course this only makes sense if A is of rank two. This will be so if there is significant gradient energy in more than one orientation.

3 Discussion

- Different neighborhoods around a true corner location will each yield different estimates of $\mathbf{x}_o$. Keep the one from the neighborhood whose center is closest to $\mathbf{x}_o$.
- The matrix A is known as the “windowed image second moment matrix” or just “second moment matrix.” The same matrix appears in Lucas and Kanade’s method of recovering optical flow.
- Note the similarity between A and the covariance matrix for a collection of n gradient vectors.
- The use of the second moment matrix (viz. its eigenvalues and eigenvectors) for local feature analysis goes by many names: Plessey-Harris operator, Förstner corner detector, orientation tensor, local structure tensor, etc. (So it must be important!)

- By substituting normal lines for tangents, the above least-squares formulation can be modified in a straightforward way to find the center of circular features instead of corners.
- A spatial weighting function $w(\mathbf{x}')$ (e.g. a Gaussian) is sometimes used in the above integrals to give more weight to pixels near the center of the neighborhood.

4 References

1. Förstner and Gülch 1987
2. Malik CS280 notes 1995