Pumping Lemma

If A is a regular language, then there is a no. p *(pumping length)* **where, if s is any string in A of length at least p, s may be divided into three pieces x,y,z, s = xyz, such that all of the following hold:**

1. **for each $i \geq 0$, xy^iz is in A**
2. **$|y| > 0$**
3. **$|xy| \leq p$**

Condition 1 lets us "pump out'' elements in A

Note that x or z can be ε , but y cannot be (by 2)

(Without 2, the lemma is trivially true, with y = ε)

Condition 3 assures us we can make x and y small, if needed.

Alternating Quantifers in the Pumping Lemma

1. **For each regular language L**

   (Choice of L)

2. **There exists a pumping length p for L**

   (given p by lemma)

3. **For every string s in L of length $\geq p$**

   (Choice of s)

4. **There exists x, y, z, with s = xyz, $|y| > 0$ and $|xy| \leq p$.**

   (lengths y and xy restricted)

5. **For each $i \geq 0$ xy^iz in L.**

   (Choose an i that leads to contradiction)

Pumping Lemma Example

L = $\{ 0^n 1^n \mid n \geq 0 \}$ is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.
Let s = $0^p 1^p$ in L. Note that |s| > p, so s = xyz with
1. for each i $\geq$ 0, $xy^i z$ is in L
2. |y| > 0
(Don't use condition 3)

Case 1: **y = 0 ... 0**

Then xyyz will have more 0's than 1's,

so it cannot be in L, a contradiction.

Pumping Lemma Example

Case 2: **y = 1 ... 1**

Then xyyz will have more 1's than 0's,

so it cannot be in L, a contradiction.

Case 3: **y = 0 ... 0 1 ... 1**

Then xyyz will have 0's and 1's out of order, with some 1's before 0's, a contradiction.

Since these are all possible cases, we can conclude that L is not regular.

Pumping Lemma Example, using 3.

$L = \{ 0^n 1^n \mid n \geq 0 \}$ is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.

Let $s = 0^p 1^p$ in L. Note that $|s| > p$, so s = xyz with

1. for each $i \geq 0$, $xy^i z$ is in L
2. $|y| > 0$
3. $|xy| \leq p$

It must be the case that y = 0 ... 0, since xy is shorter than p.

But then xyyz will have more 0's than 1's,

so it cannot be in L, a contradiction.

Proving Languages Non-Regular

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. Choose a string s in L of length $\geq$ p.
   (Note that not every string in L will work!)
4. Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 and 3.
   Show for each case that there is an $i \geq 0$ with $xy^i z$ not in L.
5. This provides a contradiction to the assumption of the pumping lemma, and to L being regular.

Examples: Showing Language L Non-Regular

L = {w | w has an equal no of 0 and 1's} *(1.39 p.80)*

1. **Assume L is regular. Then the Pumping Lemma holds.**
2. **Let p be the pumping length for L given by the lemma.**
3. **We choose s= 0^p1^p (in L of length ≥ p).**
4. **Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 (|y| > 0) and condition 3 (|xy| ≤ p).**
   For this s, it must be that $y = 0^k$; this is the only case. We choose i=2: xy^2z will have more 0's than 1's, and so cannot be in L.
5. **Contradiction.**

Examples: Showing Language L Non-Regular

L = {ww | w in {0, 1} * } *(1.40 p.81)*

1. **Assume L is regular. Then the Pumping Lemma holds.**
2. **Let p be the pumping length for L given by the lemma.**
3. **We choose s= 0^p10^p1 (in L of length ≥ p).**
4. **Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 (|y| > 0) and condition 3 (|xy| ≤ p).**
   For this s, it must be that $y = 0^k$; this is the only case. We choose i=2: xy^2z will have more 0's before the first 1 than the second 1; not in L.
5. **Contradiction.**

Examples: Showing Language L Non-Regular

$L = \{0^i 1^j \mid i > j\}$ *(1.42 p.82)*

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose $s = 0^{p+1} 1^p$ (in L of length $\geq$ p).
4. Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 ($|y| > 0$) and condition 3 ($|xy| \leq p$).
   For this s, it must be that $y = 0^k$; this is the only case. We choose i=0: xz does not have more 0's than 1's, and so cannot be in L. *(i = 2?)*
5. Contradiction.

Another Example

Show $\{ a^n b^n c^n \mid n \geq 0\}$ is not regular.

Show $\{ a^n b a^m b a^{n+m} \mid n, m \geq 1\}$ is not regular.

Pumping Lemma Proof

Proof Idea: Let M be a DFA recognizing A.
Let p = the number of states in M.

Consider any string s in A of length $\geq$ *p (if none, done!)*

Suppose s = s1 s2 ...sn.

Consider the sequence of states q0 q1 q2 ...qn go through on s.

$q0 \xrightarrow{s1} q1 \xrightarrow{s2} \ldots\ldots\ldots \xrightarrow{sn} qn$ *(qn final)*

There are n + 1 states, $n \geq p$.

But p is the number of states of M, so there must be a repetition. (Pigeonhole principle) Pick the first one.

$q0 \xrightarrow{s1} \xrightarrow{sr-1} qr \ldots\ldots qr \xrightarrow{} \xrightarrow{sn} qn$

x y z

Proof of the Pumping Lemma

Let M = (Q,Σ,δ, q_1, F) be a DFA recognizing A.
Let p = |Q|, the number of states of M.
Let s=s1...sn be any string of length n, n ≥ p.
Let r1 ... rn+1 be the sequence of states that M enters when processing s, so ri+1 = δ(ri, si) for 1 ≤ i ≤ n. In the first p+1 elements of this sequence r1 ... rn+1, there must be 2 states that are the same, by the pigeonhole principle.
Call them rj and rl.
We let x = s1...sj-1, y = sj...sl-1, z = sl...sn .
M must accept $x y^i z$ for all i ≥ 0
j is not equal to l, so |y| > 0
l is less than p+1, so |xy| ≤ p, satisfying all cond.