

Pumping Lemma

If A is a regular language, then there is a no. p
(pumping length) where, if s is any string in A of length
at least p , s may be divided into three pieces x,y,z ,
 $s = xyz$, such that all of the following hold:

1. for each $i \geq 0$, $xy^i z$ is in A
2. $|y| > 0$
3. $|xy| \leq p$

Condition 1 lets us "pump out" elements in A

Note that x or z can be ϵ , but y cannot be (by 2)

(Without 2, the lemma is trivially true, with $y = \epsilon$)

Condition 3 assures us we can make x and y small, if needed.

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Alternating Quantifiers in the Pumping Lemma

1. For each regular language L

(Choice of L)

2. There exists a pumping length p for L

(given p by lemma)

3. For every string s in L of length $\geq p$

(Choice of s)

4. There exists x, y, z , with $s = xyz$, $|y| > 0$ and
 $|xy| \leq p$.

(lengths y and xy restricted)

5. For each $i \geq 0$ $xy^i z$ in L .

(Choose an i that leads to contradiction)

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Pumping Lemma Example

$L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.

Let $s = 0^p 1^p$ in L . Note that $|s| > p$, so $s = xyz$ with

1. for each $i \geq 0$, $xy^i z$ is in L
2. $|y| > 0$

(Don't use condition 3)

Case 1: $y = 0 \dots 0$

Then $xyyz$ will have more 0's than 1's,
so it cannot be in L , a contradiction.

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Pumping Lemma Example

Case 2: $y = 1 \dots 1$

Then $xyyz$ will have more 1's than 0's,
so it cannot be in L , a contradiction.

Case 3: $y = 0 \dots 0 1 \dots 1$

Then $xyyz$ will have 0's and 1's out of order,
with some 1's before 0's, a contradiction.

Since these are all possible cases, we can
conclude that L is not regular.

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Pumping Lemma Example, using 3.

$L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.

Let $s = 0^p 1^p$ in L . Note that $|s| > p$, so $s = xyz$ with

1. for each $i \geq 0$, $xy^i z$ is in L
2. $|y| > 0$
3. $|xy| \leq p$

It must be the case that $y = 0 \dots 0$, since xy is shorter than p .

But then $xyyz$ will have more 0's than 1's,
so it cannot be in L , a contradiction.

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Proving Languages Non-Regular

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. Choose a string s in L of length $\geq p$.
(Note that not every string in L will work!)
4. Consider all cases s can be divided into x, y, z , $s = xyz$, satisfying conditions 2 and 3.
Show for each case that there is an $i \geq 0$ with $xy^i z$ not in L .
5. This provides a contradiction to the assumption of the pumping lemma, and to L being regular.

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Examples: Showing Language L Non-Regular

$L = \{w \mid w \text{ has an equal no of 0 and 1's}\}$ (1.39 p.80)

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose $s = 0^p 1^p$ (in L of length $\geq p$).
4. Consider all cases s can be divided into x, y, z, $s = xyz$, satisfying conditions 2 ($|y| > 0$) and condition 3 ($|xy| \leq p$).
For this s, it must be that $y = 0^k$; this is the only case. We choose $i=2$: xy^2z will have more 0's than 1's, and so cannot be in L.
5. Contradiction.

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Examples: Showing Language L Non-Regular

$L = \{ww \mid w \text{ in } \{0, 1\}^* \}$ (1.40 p.81)

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose $s = 0^p 10^p 1$ (in L of length $\geq p$).
4. Consider all cases s can be divided into x, y, z, $s = xyz$, satisfying conditions 2 ($|y| > 0$) and condition 3 ($|xy| \leq p$).
For this s, it must be that $y = 0^k$; this is the only case. We choose $i=2$: xy^2z will have more 0's before the first 1 than the second 1; not in L.
5. Contradiction.

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Examples: Showing Language L Non-Regular

$$L = \{0^i 1^j \mid i > j\} \quad (1.42 \text{ p.82})$$

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose $s = 0^{p+1} 1^p$ (in L of length $\geq p$).
4. Consider all cases s can be divided into x, y, z, $s = xyz$, satisfying conditions 2 ($|y| > 0$) and condition 3 ($|xy| \leq p$).
For this s, it must be that $y = 0^k$; this is the only case. We choose $i=0$: xz does not have more 0's than 1's, and so cannot be in L. ($i = 2$?)
5. Contradiction.

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Another Example

Show $\{a^n b^n c^n \mid n \geq 0\}$ is not regular.

Show $\{a^n b^m a^{n+m} \mid n, m \geq 1\}$ is not regular.

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Pumping Lemma Proof

Proof Idea: Let M be a DFA recognizing A .

Let p = the number of states in M .

Consider any string s in A of length $\geq p$ (if none, done!)

Suppose $s = s_1 s_2 \dots s_n$.

Consider the sequence of states $q_0 q_1 q_2 \dots q_n$ go through on s .

$q_0 \xrightarrow{s_1} q_1 \xrightarrow{s_2} \dots \xrightarrow{s_n} q_n$ (q_n final)

There are $n + 1$ states, $n \geq p$.

But p is the number of states of M , so there must be a repetition. (Pigeonhole principle) Pick the first one.

$$\begin{array}{ccccccc}
 q_0 & \xrightarrow{\quad} & & \xrightarrow{\quad} & q_r & \dots & q_r \xrightarrow{\quad} & \dots & q_n \\
 & \xrightarrow{s_1} & & \xrightarrow{s_{r-1}} & & \dots & & & \xrightarrow{s_n} \\
 & \underline{\hspace{1.5cm}} & & \underline{\hspace{1.5cm}} & & \underline{\hspace{1.5cm}} & & & \\
 & \mathbf{x} & & \mathbf{y} & & & & & \mathbf{z}
 \end{array}$$

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Proof of the Pumping Lemma

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognizing A .

Let $p = |Q|$, the number of states of M .

Let $s = s_1 \dots s_n$ be any string of length n , $n \geq p$.

Let $r_1 \dots r_{n+1}$ be the sequence of states that M enters when processing s , so $r_{i+1} = \delta(r_i, s_i)$ for $1 \leq i \leq n$. In the first $p+1$ elements of this

sequence $r_1 \dots r_{n+1}$, there must be 2 states that are the same, by the pigeonhole principle.

Call them r_j and r_l .

We let $x = s_1 \dots s_{j-1}$, $y = s_j \dots s_{l-1}$, $z = s_l \dots s_n$.

M must accept $x y^i z$ for all $i \geq 0$

j is not equal to l , so $|y| > 0$

l is less than $p+1$, so $|xy| \leq p$, satisfying all cond.

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