Multiple tapes

$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L,R\}^k$

Lemma: If L is accepted by a multitape TM, then L is accepted by a single tape TM.

Proof idea: use tracks, with special markers for each tape head

w	x	y				
a	b					
	m	n				

M' will simulate M by making multiple passes over tape, 1 for each track. First, it needs to determine all the symbols being read, in order to determine the next move. So k symbols must be stored in state. Then, once the next move is determined, it must update each track with new symbol and new tape head position.

1

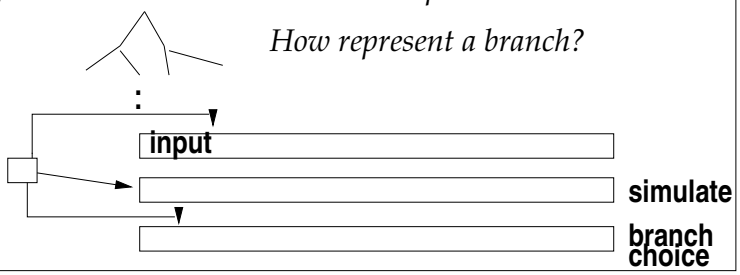
Nondeterministic TM's

$\delta : Q \times \Gamma \rightarrow P(Q \times \Gamma \times \{L,R\})$

Lemma: If L is accepted by a nondet. TM N, then L is accepted by some deterministic TM D.

Proof idea: D will simulate N by trying all possible branches of N's computation on input w. If D finds an accept state on some branch, it accepts. OW, D does not accept.

QU: How should D search the computation tree??



2

Robustness of TM's

Is this surprising??
TM's ⟷ "Idealized" programming languages

Programming languages, if sufficiently general, can do the same computations

Ex. C, Lisp, Java

TM's make formal our informal notion of algorithm

Church-Turing Thesis: The Turing machine model (and any reasonable variant of it) embodies our informal notion of algorithm.

Can't be proved!

From now on, TM ⟷ Algorithms

Decidable Problems

Represent problem using language

Problem: Is w accepted by DFA B?

A_{DFA} = *{<B,w> | B is a DFA that accepts w}*

< > *denotes encoding as string*

Language decidable ⟷ Problem decidable

Th. A_{DFA} is a decidable language.

Proof idea: Decider M will simulate D on input w.

If D would accept w, then M accepts; D rejects, M rejects.

q0q1q2q3 # 01# 01XY␣# <table># q0#q2q3##0111
0111 # q0
▲

Decidable Problems

A_{NFA} = {<B,w> | B is an NFA, B accepts w}

Th. A_{NFA} is decidable.

Proof idea: Define N to use TM M (last th) as subroutine.

1. *Convert NFA B to DFA D.*
2. *Run TM M on input <D,w>*
3. *If M accepts, then N accepts; M rejects, N rejects.*

q0 q1q2q3 # 01# 01XY␣# <nd.table>#....
{q0} {q0 q1} .....#.....# <det. table>#

Decidable Problems

E_{DFA} = {<B> | B is a DFA, L(B) is nonempty}

Th. E_{DFA} is decidable.

Lemma: For DFA B, L(B) is nonempty iff B accepts a string of length at most |Q|.

← **Trivial.**

→ **Suppose B accepts some word. Let w be a shortest word accepted by B. If |w| is ≤ |Q|, then we are done. So suppose |w| > |Q|. Consider the sequence of states of B on w.**

Decidable Problems

w = w1 w2 w3 .... wm, m > |Q|

q0 →(w1) q1 →(w2) q2 ..... →(wm) qm

Since there are more that |Q| states in this sequence, there must be a repetition of states, by the pigeonhole principle. But then we can take a snippet z out of w, and get a shorter string that is accepted by B. Contradiction. Therefore w is of length ≤ |Q|.

Back to Theorem:

Decider D will, on input <B>, simulate B on inputs of length 0 to |Q|. If B accepts any string of that length, D accepts. If no such string is found, D rejects. By the Lemma, D accepts E_{DFA} ·

Decidable Problems

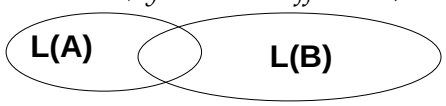
EQ_{DFA} = {<A,B> | A,B are DFA, and L(A) = L(B) }

Th: EQ_{DFA} is decidable.

Proof: We construct a DFA C with

$L(C) = (L(A) \cap \overline{L(B)}) \cup (\overline{L(A)} \cap L(B))$

(symmetric difference)



C can be constructed by TM M using algorithms we developed for closure under $\cup$, $\cap$, $\overline{\ }$

Since L(C) is empty iff. L(A) = L(B), we can use previous theorem for deciding L(C) nonempty.

Decidable Problems for CFG's

A_{CFG} = {<G,w> | G is a CFG, G generates w}

Th. A_{CFG} is decidable.

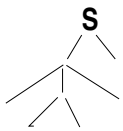
Proof idea: TM could start with start var. S and try all derivations. If w is not in L(G), there will be none!
So trying all derivations to see if get w would not yield decider.
To define a decider, we first put G in Chomsky normal form.
Then we can bound the number of steps in a derivation TM needs to try to $2|w|-1$. QU: WHY?
The decider D will then list all derivations of length 0 to $2|w|-1$. If any derivation generates w, D accepts; ow, reject.

Decidable Problems on CFG's

E_{CFG} = {<G> | G is a CFG, L(B) is nonempty}

Th. E_{CFG} is decidable.

Proof idea: Can't try all strings (won't be decidable).
Consider any parse tree for a string w.



If parse tree has path of length $> |V|$, then there is a repetition of variables along that path, and we can produce a shorter parse tree. We can repeat this process so that there is a parse tree with no path of length $> |V|$. Therefore, if G generates any string, it generates a string whose parse tree has no path $> |V|$.

Decidable Problems on CFG's

Th. E_{CFG} is decidable. ***(continued)***

A decider D for this problem will generate a collection C of parse trees on its tape. C will initially contain **S** *. D repeatedly adds to C any tree that can be obtained from one already in C by applying a single rule, such that*

1. *the new tree is not in C*
2. *the new tree does not have any path of len.* $> |V|$

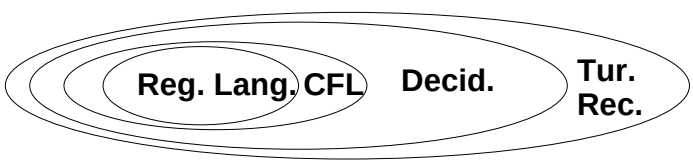
Since there are only a finite number of trees of fixed length, the TM D will eventually complete C. L(G) is nonempty iff at least one tree in C has only terminals as leaves.

Decidable Problems for CFG's

EQ_{CFG} = {<G,H> | G,H are CFG, and L(G) = L(H) }

EQ_{CFG} decidable??

Can't use same idea we used for DFA's (symmetric diff.) because CFL's are NOT closed under complement.

Turns out EQ_{CFG} is NOT decidable. We can't prove it yet.

Th. Every CFL is decidable. (HW 3)