Practice Problem Answers - recurrences and asymptotic notation

These are just the answers, so that you can verify for yourself that you got the right answer.

Problem 2-2 Table entries in left to right order for each row:

a. yes, yes, no, no, no

b. yes, yes, no, no, no

c. no, no, no, no, no

d. no, no, yes, yes, no

e. yes, no, yes, no, yes

f. yes, no, yes, no, yes

Problem 4-1 **a.** $T(n) = \Theta(n^3)$ (use case 3 of the master theorem)

b. $T(n) = \Theta(n)$ (use case 3 of the master theorem)

c. $T(n) = \Theta(n^2 \lg n)$ (use case 2 of the master theorem)

d. $T(n) = \Theta(n^2)$ (use case 3 of the master theorem) of the master theorem)

e. $T(n) = \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81})$ (use case 1 of the master theorem)

f. $T(n) = \Theta(n^{1/2} \lg(n))$ (use case 2 of the master theorem)

g. $T(n) = \Theta(n^2)$ (iteration method; see page 156 for details)

h. $T(n) = \Theta(\lg(\lg(n)))$. Note that you might iterate the recurrence to figure out that this is the proper bound; the constant 1 is added each time we iterate, so that we get $T(n) = T(n^{(1/2^k)}) + k$ for larger and larger k. This will stop once $1/2^k$ gets down to about $\frac{1}{\lg(n)}$, since at that point $n^{(1/2^k)}$ is about 1. Solving $2^k = \lg(n)$ gives $k = \lg(\lg(n))$, which suggests that we'll add 1 about $\lg(\lg(n))$ times before the problem size gets too small to crank the recurrence anymore.

Use the substitution method to prove this bound is correct. First show that $T(n) \leq c\lg(\lg(n))$ for $n > 2$. For the base case, pick c large so that this is satisfied for $2 < n \leq 4$ (we need this so that any recurrences that start with larger n (that is, $n > 4$ so that $n^{1/2} > 2$) will eventually depend on this base case, rather than falling through to some smaller n for which we don't have any bound). Now, assume this bound holds for values less than n, and show for n. Substituting into the recurrence, we get:

$$T(n) \leq c\lg(\lg(n^{1/2})) + 1$$

$$T(n) \leq c\lg(\frac{1}{2}\lg(n)) + 1$$

$$T(n) \leq c\lg(\frac{1}{2}) + c\lg(\lg(n)) + 1$$

$$T(n) \leq c\lg(\lg(n)) + (1 - c)$$

Pick $c > 1$, so that we can drop the $(1 - c)$ term and get our desired conclusion:

$$T(n) \leq c\lg(\lg(n))$$

So, $T(n) = O(\lg(\lg(n)))$.

Since we want a Θ bound, we also need a lower bound on $T(n)$. We want to show that $T(n) \geq c\lg(\lg(n))$. Similar to the above, for the base case pick c small enough so that this bound holds for $2 < n \leq 4$. Now, assume the bound holds for values less than n and show for n. Substituting into the recurrence:

$$T(n) \geq c\lg(\lg(n^{1/2})) + 1$$

$$T(n) \geq c\lg(\frac{1}{2}\lg(n)) + 1$$

$$T(n) \geq c\lg(\frac{1}{2}) + c\lg(\lg(n)) + 1$$

$$T(n) \geq c\lg(\lg(n)) + (1-c)$$

Pick $c < 1$, so that we can drop the $(1-c)$ term and get our desired conclusion:

$$T(n) \geq c\lg(\lg(n))$$

So, $T(n) = \Omega(\lg(\lg(n)))$. Combining this with the above, $T(n) = \Theta(\lg(\lg(n)))$.