# Calibrated Stereo (Part 1) 

## Computer Vision I <br> CSE 252A <br> Lecture 7

## Announcements

- Assignment 1 is due Oct 25, 11:59 PM
- Assignment 2 will be released Oct 25
- Due Nov 8, 11:59 PM


## Why Do We Have Two Eyes?



1. Redundancy - If we lose one, we're not blind
2. Larger field of view
3. Ability to recover depth for some points

## Why Do We Have Two Eyes?



## Stereo Vision



Holmes Stereoscope

An Application: Mobile Robot Navigation


Mobi, Stanford, 1987
The Stanford Cart, H. Moravec, 1979


IRIA
$-2+2-2$

## Mars Exploratory Rovers: Spirit and Opportunity, 2004



## Curiosity Rover (2012)



- Navigation cameras (Navcams) B\&W, $45^{\circ}$ field of view
- Hazard avoidance cameras (hazcams), 4 pairs, $120^{\circ}$ field of view


## Boston Dynamics



Stereo + Lidar


## Commercial Stereo Heads



## Binocular Stereopsis: Mars

 Given two images of a scene where relative locations of cameras are known, estimate depth of all common scene points.

Two images of Mars (Viking Lander)

## Matching complexity (naïve)



Input: two images that are nx n pixels

For a given point in the left image, where do we look in the right image?


For each point in left mage, there are $\mathrm{O}\left(\mathrm{n}^{2}\right)$ possible matching points in right image.

With $n^{2}$ pixels in left image, complexity of matching is $\mathrm{O}\left(\mathrm{n}^{4}\right)$

Can we do better?

## Epipolar Geometry Terminology



- Baseline: line connecting camera centers (of projection) $\mathbf{C}$ and $\mathbf{C}^{\prime}$
- Epipoles (e, e'): Two intersection points of baseline with image planes
- Epipolar Plane: Any plane that contains the baseline
- Epipolar Lines (l, l'): Pair of lines from intersection of an epipolar plane with the two image planes


## Family of Epipolar Planes



- Epipolar Plane: Any plane that contains the baseline
- The set of epipolar planes is a family of all planes passing through the baseline and can be parameterized by the angle about baseline


## Epipolar matching



- Potential matches for $\mathbf{p}$ have to lie on the corresponding epipolar line $\mathbf{l}{ }^{\prime}$
- Epipolar line l' passes through epipole $\mathbf{e}^{\prime}$, the intersection of the baseline with the image plane
- Potential matches for $\mathbf{p}$ ' have to lie on the corresponding epipolar line $\mathbf{I}$


## Epipolar matching complexity



Using epipolar matching, complexity is reduced from $\mathrm{O}\left(\mathrm{n}^{4}\right)$ to $\mathrm{O}\left(\mathrm{n}^{3}\right)$. Why?

- There are $\mathrm{n}^{2}$ points in the left image
- For each point in the left image, all candidate matches are on an epipolar line in the right image, and the length of the epipolar line is $\mathrm{O}(\mathrm{n})$
- Therefore, match complexity is $\mathrm{O}\left(\mathrm{n}^{2 *} \mathrm{n}\right)=\mathrm{O}\left(\mathrm{n}^{3}\right)$


## Stereo Vision Outline

- Offline
- Calibration of stereo cameras
- Online

1. Acquire stereo images
2. Epipolar rectify stereo images
3. Establish correspondence
4. Estimate depth

## Calibration of stereo cameras

1. From images of known calibration fixture, determine intrinsic parameters $\mathbf{K}_{1}, \mathbf{K}_{2}$ and extrinsic relation of two cameras $\mathbf{R}_{1}, \mathbf{t}_{1}$ and $\mathbf{R}_{2}, \mathbf{t}_{2}$
2. Compute the relative rotation $\mathbf{R}$ and translation $\mathbf{t}$ of the two cameras from $\mathbf{R}_{1}, \mathbf{t}_{1}$ and $\mathbf{R}_{2}, \mathbf{t}_{2}$
3. Compute the essential matrix $\mathbf{E}$


## Camera calibration



- Given $n$ points $\mathbf{P}_{1}, \ldots, \mathbf{P}_{\mathrm{n}}$ with known 3-D position and their pixel coordinates $\mathbf{x}_{1}, \ldots, \mathbf{x}_{\mathrm{n}}$, estimate intrinsic $\mathbf{K}$ and extrinsic camera parameters and lens distortion parameters
- See textbook for details
- Camera Calibration Toolbox for Matlab (Bouguet) http://www.vision.caltech.edu/bouguetj/calib_doc/


## Compute the rotation and translation of the second camera relative to the first one

$$
\begin{aligned}
\mathbf{x}_{1}=\mathrm{K}_{1}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
\mathrm{R}_{1} & \mathbf{t}_{1} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} & \mathbf{x}_{2}=\mathrm{K}_{2}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
\mathrm{R}_{2} & \mathbf{t}_{2} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} \\
\mathbf{x}_{1}=\mathrm{K}_{1}[\mathrm{I} \mid \mathbf{0}] \mathbf{X}_{\mathrm{cam}, 1} & \mathbf{x}_{2}=\mathrm{K}_{2}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
\mathrm{R}_{2} & \mathbf{t}_{2} \\
\mathbf{0}^{\top} & 1
\end{array}\right]\left[\begin{array}{cc}
\mathrm{R}_{1} & \mathbf{t}_{1} \\
\mathbf{0}^{\top} & 1
\end{array}\right]^{-1} \mathbf{X}_{\mathrm{cam}, 1} \\
\text { where } \mathbf{X}_{\mathrm{cam}, 1}=\left[\begin{array}{cc}
\mathrm{R}_{1} & \mathbf{t}_{1} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} & \mathbf{x}_{2}=\mathrm{K}_{2}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
\mathrm{R}_{2} & \mathbf{t}_{2} \\
\mathbf{0}^{\top} & 1
\end{array}\right]\left[\begin{array}{cc}
\mathrm{R}_{1}^{\top} & -\mathrm{R}_{1}^{\top} \mathbf{t}_{1} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X}_{\mathrm{cam}, 1} \\
{\left[\begin{array}{cc}
\mathrm{R}_{1} & \mathbf{t}_{1} \\
\mathbf{0}^{\top} & 1
\end{array}\right]^{-1} \mathbf{X}_{\mathrm{cam}, 1}=\mathbf{X} } & \mathbf{x}_{2}=\mathrm{K}_{2}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
\mathrm{R}_{2} \mathrm{R}_{1}^{\top} & \mathbf{t}_{2}-\mathrm{R}_{2} \mathrm{R}_{1}^{\top} \mathbf{t}_{1} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X}_{\mathrm{cam}, 1} \\
& \mathbf{x}_{2}=\mathrm{K}_{2}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
\mathrm{R} & \mathbf{t} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X}_{\mathrm{cam}, 1}
\end{aligned}
$$

$$
\text { where } \begin{aligned}
& \mathrm{R}=\mathrm{R}_{2} \mathrm{R}_{1}^{\top} \text { and } \\
& \mathbf{t}=\mathbf{t}_{2}-\mathrm{R}_{2} \mathrm{R}_{1}^{\top} \mathbf{t}_{1}
\end{aligned}
$$

## Image points

- Image points in pixel coordinates

$$
\begin{aligned}
& \mathbf{x}=K[I \mid \mathbf{0}]\left[\begin{array}{cc}
R & \mathbf{t} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} \\
& \mathbf{x}=K[R \mid \mathbf{t}] \mathbf{X}
\end{aligned}
$$

- Image points in normalized coordinates

$$
\begin{aligned}
\mathbf{x} & =\mathrm{K}[\mathrm{R} \mid \mathbf{t}] \mathbf{X} \\
\mathrm{K}^{-1} \mathbf{x} & =[\mathrm{R} \mid \mathbf{t}] \mathbf{X} \\
\hat{\mathbf{x}} & =[\mathrm{R} \mid \mathbf{t}] \mathbf{X} \text { where } \hat{\mathbf{x}}=\mathrm{K}^{-1} \mathbf{x}
\end{aligned}
$$

## Image points in normalized coordinates

$$
\begin{aligned}
\mathbf{x} & =\mathrm{K}[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
R & \mathbf{t} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} \\
\mathrm{K}^{-1} \mathbf{X} & =[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{cc}
R & \mathbf{t} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} \\
\hat{\mathbf{x}} & =[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{ll}
R & \mathbf{t} \\
\mathbf{0}^{\top} & 1
\end{array}\right] \mathbf{X} \text { where } \hat{\mathbf{x}}=\mathrm{K}^{-1} \mathbf{X} \\
\hat{\mathbf{x}} & =[\mathrm{I} \mid \mathbf{0}] \mathbf{X}_{\mathrm{cam}} \\
\hat{\mathbf{x}} & =[\mathrm{I} \mid \mathbf{0}]\left[\begin{array}{c}
\tilde{\mathbf{X}}_{\text {cam }} \\
1
\end{array}\right] \\
\hat{\mathbf{x}} & =\tilde{\mathbf{X}}_{\text {cam }} \text { (up to nonzero scale) }
\end{aligned}
$$

## The essential matrix

3D point $\tilde{\mathbf{X}}^{\prime}=\lambda^{\prime} \hat{\mathbf{x}}^{\prime}$ in the second camera coordinate frame.
3 D point $\tilde{\mathbf{X}}=\lambda \hat{\mathbf{x}}$ in the first camera coordinate frame.
Map $\tilde{\mathbf{X}}$ from first camera coordinate frame to second camera coordinate frame.

$$
\begin{aligned}
& \tilde{\mathbf{X}}^{\prime}=R \tilde{\mathbf{X}}+\mathbf{t} \\
& \tilde{\mathbf{X}}^{\prime}=R(\lambda \hat{\mathbf{x}})+\mathbf{t} \\
& \tilde{\mathbf{X}}^{\prime}=\lambda R \hat{\mathbf{x}}+\mathbf{t}
\end{aligned}
$$



## The essential matrix

3D point $\tilde{\mathbf{X}}^{\prime}=\lambda^{\prime} \hat{\mathbf{x}}^{\prime}$ in the second camera coordinate frame.
3D point $\tilde{\mathbf{X}}=\lambda \hat{\mathbf{x}}$ in the first camera coordinate frame.
Map $\tilde{\mathbf{X}}$ from first camera coordinate frame to second camera coordinate frame.

$$
\begin{aligned}
& \tilde{\mathbf{X}}^{\prime}=\mathrm{R} \tilde{\mathbf{X}}+\mathbf{t} \\
& \tilde{\mathbf{X}}^{\prime}=\mathrm{R}(\lambda \hat{\mathbf{x}})+\mathbf{t} \\
& \tilde{\mathbf{X}}^{\prime}=\lambda \mathrm{R} \hat{\mathbf{x}}+\mathrm{t} \\
& \lambda \mathrm{R} \hat{\mathbf{x}}+\mathrm{t}=\lambda^{\prime} \hat{\mathbf{x}}^{\prime} \\
& {[\mathbf{t}]_{\times}(\lambda R \hat{\mathbf{x}}+\mathbf{t})=[\mathbf{t}]_{\times}\left(\lambda^{\prime} \hat{\mathbf{x}}^{\prime}\right)} \\
& \lambda[\mathbf{t}]_{\times} \mathrm{R} \hat{\mathbf{x}}=\lambda^{\prime}[\mathbf{t}]_{\times} \hat{\mathbf{x}}^{\prime} \\
& \hat{\mathbf{x}}^{\prime \top}\left(\lambda[\mathbf{t}]_{\times} \mathrm{R} \hat{\mathbf{x}}\right)=\hat{\mathbf{x}}^{\prime \top}\left(\lambda^{\prime}[\mathbf{t}]_{\times} \hat{\mathbf{x}}^{\prime}\right) \\
& \lambda \hat{\mathbf{x}}^{\prime \top}[\mathbf{t}]_{\times} \mathrm{R} \hat{\mathbf{x}}=0 \\
& \hat{\mathbf{x}}^{\top}[\mathbf{t}]_{\times} \mathrm{R} \hat{\mathbf{x}}=0 \\
& \text { The epipolar constraint } \\
& \hat{\mathbf{x}}^{\prime \top} \mathrm{E} \hat{\mathbf{x}}=0 \text { where } \mathrm{E}=[\mathrm{t}]_{\times R} \mathrm{R}
\end{aligned}
$$

## Cross product using a skew symmetric matrix

- The cross product $\mathbf{a} \times \mathbf{b}$ of two 3-vectors $\mathbf{a}=\left(a_{1}, a_{2}, a_{3}\right)^{\top}$ and $\mathbf{b}=\left(b_{1}, b_{2}, b_{3}\right)^{\top}$ can be expressed a matrix-vector product $[\mathbf{a}]_{\times} \mathbf{b}$, where $[\mathrm{a}]_{\times}$is the $3 \times 3$ skew symmetric matrix

$$
[\mathbf{a}]_{\times}=\left[\begin{array}{ccc}
0 & -a_{3} & a_{2} \\
a_{3} & 0 & -a_{1} \\
-a_{2} & a_{1} & 0
\end{array}\right]
$$

- A matrix $\mathbf{S}$ is skew symmetric if and only if $\mathbf{S}=-\mathbf{S}^{\mathrm{T}}$
- The determinant of a skew symmetric matrix is 0


## The essential matrix

- Maps a point (in normalized coordinates) in the first image to its corresponding epipolar line (in normalized coordinates) in the second image

$$
\hat{\ell}^{\prime}=E \hat{x}
$$

- The epipolar line passes through the corresponding point in the second image

$$
\begin{aligned}
\hat{\mathbf{x}}^{\prime \top} \hat{\ell}^{\prime} & =0 \\
\hat{\mathbf{x}}^{\prime \top} E \hat{\mathbf{x}} & =0 \quad \text { The epipolar constraint }
\end{aligned}
$$

- Every epipolar line passes through the epipole

$$
\hat{\mathbf{e}}^{\top} \top \hat{\ell}^{\prime}=0
$$

## The essential matrix

- Maps a point (in normalized coordinates) in the second image to its corresponding epipolar line (in normalized coordinates) in the first image

$$
\begin{aligned}
\hat{\ell} & =\mathrm{E}^{\top} \hat{\mathbf{x}}^{\prime} \\
\hat{\ell}^{\top} & =\hat{\mathbf{x}}^{\prime \top} \mathrm{E}
\end{aligned}
$$

- The epipolar line passes through the corresponding point in the first image

$$
\begin{aligned}
\hat{\ell}^{\top} \hat{\mathbf{x}} & =0 \\
\hat{\mathbf{x}}^{\prime \top} E \hat{\mathbf{x}} & =0 \quad \text { The epipolar constraint }
\end{aligned}
$$

- Every epipolar line passes through the epipole

$$
\hat{\ell}^{\top} \hat{\mathbf{e}}=0
$$

## Example of using the essential matrix



## Another example: converging cameras



## Another example: second camera in front of first camera



# Epipolar rectify stereo images 

- Epipolar geometry reduces matching complexity from $\mathrm{O}\left(\mathrm{n}^{4}\right)$ to $\mathrm{O}\left(\mathrm{n}^{3}\right)$
- But matching requires comparing points across pairs of epipolar lines which may have arbitrary orientation. That can be costly to index.
- Is there a more convenient epipolar geometry?


Slanted epipolar lines


Horizontal, row aligned epipolar lines

## Cameras with a convenient epipolar geometry

- When two cameras have parallel optical axes and these axis are orthogonal to the baseline, the epipolar lines are parallel
- When rows of the two images are parallel to the baseline, the epipolar lines are horizontal rows of the two images



## Cameras with a convenient epipolar geometry

- When two cameras have parallel optical axes and these axis are orthogonal to the baseline, the epipolar lines are parallel
- When rows of the two images are parallel to the baseline, the epipolar lines are horizontal rows of the two images

What if stereo geometry is not convenient? Rectification: Given a pair of images, transform both images so that epipolar lines are image rows


## Epipolar rectification

- Given a pair of images, transform both images so that epipolar lines are image rows
- Create pair of virtual cameras
- The virtual cameras have the same camera centers as real cameras
- Both virtual cameras have the same:
- Camera rotation matrix R

- Camera calibration matrix K


## Epipolar rectification

- Given calibrated stereo cameras (i.e., $\mathbf{K}_{1}, \mathbf{R}_{1}$, $\mathbf{t}_{1}, \mathbf{K}_{2}, \mathbf{R}_{2}, \mathbf{t}_{2}$ ) determine the (same) rotation matrix $\mathbf{R}$ and (same) calibration matrix $\mathbf{K}$ of the virtual cameras
- To minimize image distortion:
- For the calibration matrix
- For principal point, set $x_{0}$ and $y_{0}$ to average of input $x_{0}$ and $y_{0}$ values, respectively
- Set two focal length parameters to average of all four input focal length parameters (results in square pixels)
- Set skew to 0
- For the rotation matrix $\mathbf{R}$, interpolate halfway between the two 3D rotations embodied by $\mathbf{R}_{1}$ and $\mathbf{R}_{2}$


## Epipolar rectification

- For the rotation matrix $\mathbf{R}$, interpolate halfway between the two 3D rotations embodied by $\mathbf{R}_{1}$ and $\mathbf{R}_{2}$
- Rotating a camera does not change its center $\mathbf{C}$, but does change its translation $\mathbf{t}$

$$
\begin{aligned}
\tilde{\mathbf{0}}_{\mathrm{cam}} & =\mathrm{R} \tilde{\mathbf{C}}+\mathrm{t} \\
-\mathrm{R} \tilde{\mathrm{C}} & =\mathrm{t} \\
\tilde{\mathrm{C}} & =-\mathrm{R}^{\top} \mathbf{t}
\end{aligned}
$$

$$
\mathbf{x}=\mathrm{K}[\mathrm{R} \mid \mathrm{t}] \mathbf{X}
$$

$$
\begin{aligned}
\tilde{\mathbf{C}} & =-\mathrm{R}_{\text {virtual }}^{\top} \mathbf{t}_{\text {virtual }} \\
\tilde{\mathbf{C}} & =-\mathrm{R}_{\text {real }}^{\top} \mathbf{t}_{\text {real }} \\
-\mathrm{R}_{\text {virtual }}^{\top} \mathbf{t}_{\text {virtual }} & =-\mathrm{R}_{\text {real }}^{\top} \mathbf{t}_{\text {real }} \\
\mathbf{t}_{\text {virtual }} & =\mathrm{R}_{\text {virtual }} \mathrm{R}_{\text {real }}^{\top} \mathbf{t}_{\text {real }}
\end{aligned}
$$

$$
\mathbf{x}=K[R \mid-R \tilde{\mathbf{C}}] \mathbf{X}
$$

$$
\mathbf{x}=\operatorname{KR}[I \mid-\tilde{\mathbf{C}}] \mathbf{X}
$$

## Rectification transformation matrices

- Transformation from image acquired by real camera to image acquired by virtual camera

$$
\begin{gathered}
\mathbf{x}_{\text {real }}=\mathrm{K}_{\text {real }} \mathrm{R}_{\text {real }}[\mathrm{I} \mid-\tilde{\mathbf{C}}] \mathbf{X} \\
\mathrm{R}_{\text {real }}^{\top} \mathrm{K}_{\text {real }}^{-1} \mathbf{x}_{\text {real }}=[\mathrm{I} \mid-\tilde{\mathbf{C}}] \mathbf{X} \\
\mathbf{x}_{\text {virtual }}=\mathrm{K}_{\text {virtual }} \mathrm{R}_{\text {virtual }}[\mathrm{I} \mid-\tilde{\mathbf{C}}] \mathbf{X} \\
\mathbf{x}_{\text {virtual }}=\mathrm{K}_{\text {virtual }} \mathrm{R}_{\text {virtual }} \mathrm{R}_{\text {real }}^{\top} \mathrm{K}_{\text {real }}^{-1} \mathbf{x}_{\text {real }} \\
\mathbf{x}_{\text {virtual }}=\mathrm{H} \mathbf{x}_{\text {real }}, \text { where } \mathrm{H}=\mathrm{K}_{\text {virtual }} \mathrm{R}_{\text {virtual }} \mathrm{R}_{\text {real }}^{\top} \mathrm{K}_{\text {real }}^{-1} \\
{\left[\begin{array}{c}
x_{\text {virtual }} \\
y_{\text {virtual }} \\
w_{\text {virtual }}
\end{array}\right]=\left[\begin{array}{lll}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{array}\right]\left[\begin{array}{l}
x_{\text {real }} \\
y_{\text {real }} \\
w_{\text {real }}
\end{array}\right]}
\end{gathered}
$$

## Rectification

Under perspective projection, the mapping from a plane to a plane is given by a 2D projective transformation (homography)


## Rectification

Under perspective projection, the mapping from a plane to a plane is given by a 2D projective transformation (homography)


## Image pair rectification

Apply projective transformation so that epipolar lines correspond to horizontal scanlines


H should map epipole e to ( $1,0,0$ ), a point at infinity on the $x$-axis H should minimize image distortion

Note that rectified images are usually not rectangular

## Rectification

Given a pair of images, transform both images so that epipolar lines are scan lines.


## Input Images

## Rectification

Given a pair of images, transform both images so that epipolar lines are scan lines.


Rectified Images
epipolar lines run parallel with the $x$-axis and are aligned between two views (no $y$ disparity)

## Epipolar rectification

- Input: Source image I and


## Rectification matrix H

- Forward "warping" method
- For each pixel coordinate $\mathbf{x}_{\text {real }}$ in source image, map it to pixel coordinate $\mathbf{x}_{\text {virtual }}$ in target image

$$
\mathbf{x}_{\mathrm{virttual}}=\mathrm{H} \mathbf{x}_{\text {real }}
$$



- Problem: there is no guarantee that every pixel in target Image will be written to
- If target image is larger than source image or target image is highly stretched, then there may be missing points that appear as speckles or lines



## Epipolar rectification

- Input: Source image I and


## Rectification matrix H

- Backward "warping" method
- For each pixel coordinate $\mathbf{x}_{\text {virtual }}$ in target image, map it to pixel coordinate $\mathbf{x}_{\text {real }}$ in source image

$$
\mathbf{x}_{\text {real }}=\mathrm{H}^{-1} \mathbf{x}_{\text {virtual }}
$$

- Interpolate pixel values in source image to
 determine pixel value in destination image
- Problem: some of the pixel coordinates in the target image map to pixel coordinates outside of the source image
- You'll solve this problem in a homework assignment by determining where the pixel coordinates of the corners of the source image map to pixel coordinates in the destination image using

$$
\mathbf{x}_{\mathrm{virttual}}=\mathrm{H} \mathbf{x}_{\mathrm{real}}
$$



## Rectification



## Rectification

- Epipolar lines



## Rectification



## Polar Rectification



Homography-based
Rectification

Polar Rectification


Alternative epipolar rectification method that minimizes pixel distortion


## Polar Rectification



Epipoles are in images (white dot on ball)


Homography-based rectification is not possible

## Features on same epipolar line



## Next Lecture

- Calibrated stereo and feature matching

