Calibrated Stereo (Part 1)

Computer Vision I
CSE 252A
Lecture 7
Announcements

• Assignment 1 is due Oct 25, 11:59 PM
• Assignment 2 will be released Oct 25
  – Due Nov 8, 11:59 PM
Why Do We Have Two Eyes?

1. Redundancy – If we lose one, we’re not blind
2. Larger field of view
3. Ability to recover depth for some points
Why Do We Have Two Eyes?

Binocular (stereo) vision enables depth estimation.

Depth information is lost in image formation.
Stereo Vision

Holmes Stereoscope
An Application: Mobile Robot Navigation

The Stanford Cart, H. Moravec, 1979

Mobi, Stanford, 1987

INRIA Mobile Robot 1990
Mars Exploratory Rovers:
Spirit and Opportunity,
2004
Curiosity Rover (2012)

- Navigation cameras (Navcams) B&W, 45° field of view
- Hazard avoidance cameras (hazcams), 4 pairs, 120° field of view
Boston Dynamics

Stereo + Lidar
Commercial Stereo Heads
Binocular Stereopsis: Mars
Given two images of a scene where relative locations of cameras are known, estimate depth of all common scene points.

Two images of Mars (Viking Lander)
Matching complexity (naïve)

Input: two images that are $n \times n$ pixels

For each point in left image, where do we look in the right image?

For each point in left image, there are $O(n^2)$ possible matching points in right image.

With $n^2$ pixels in left image, complexity of matching is $O(n^4)$

Can we do better?
Epipolar Geometry Terminology

- **Baseline**: line connecting camera centers (of projection) $C$ and $C'$
- **Epipoles** $(e, e')$: Two intersection points of baseline with image planes
- **Epipolar Plane**: Any plane that contains the baseline
- **Epipolar Lines** $(l, l')$: Pair of lines from intersection of an epipolar plane with the two image planes
Family of Epipolar Planes

- Epipolar Plane: Any plane that contains the baseline
- The set of epipolar planes is a family of all planes passing through the baseline and can be parameterized by the angle about baseline
Epipolar matching

- Potential matches for $p$ have to lie on the corresponding epipolar line $l'$.
- Epipolar line $l'$ passes through epipole $e'$, the intersection of the baseline with the image plane.
- Potential matches for $p'$ have to lie on the corresponding epipolar line $l$. 

Using epipolar matching, complexity is reduced from $O(n^4)$ to $O(n^3)$. Why?

- There are $n^2$ points in the left image
- For each point in the left image, all candidate matches are on an epipolar line in the right image, and the length of the epipolar line is $O(n)$
- Therefore, match complexity is $O(n^2 \times n) = O(n^3)$
Stereo Vision Outline

- Offline
  - Calibration of stereo cameras

- Online
  1. Acquire stereo images
  2. Epipolar rectify stereo images
  3. Establish correspondence
  4. Estimate depth
Calibration of stereo cameras

1. From images of known calibration fixture, determine intrinsic parameters $K_1, K_2$ and extrinsic relation of two cameras $R_1, t_1$ and $R_2, t_2$

2. Compute the relative rotation $R$ and translation $t$ of the two cameras from $R_1, t_1$ and $R_2, t_2$

3. Compute the essential matrix $E$
Camera calibration

- Given $n$ points $P_1, \ldots, P_n$ with known 3-D position and their pixel coordinates $x_1, \ldots, x_n$, estimate intrinsic $K$ and extrinsic camera parameters and lens distortion parameters.
- See textbook for details.
- Camera Calibration Toolbox for Matlab (Bouguet):
  http://www.vision.caltech.edu/bouguetj/calib_doc/
Compute the rotation and translation of the second camera relative to the first one

\[
x_1 = K_1[I | 0] \begin{bmatrix} R_1 & t_1 \\ 0^\top & 1 \end{bmatrix} X
\]

\[
x_1 = K_1[I | 0]X_{\text{cam,1}}
\]

where \(X_{\text{cam,1}} = \begin{bmatrix} R_1 & t_1 \\ 0^\top & 1 \end{bmatrix} X\)

\[
\begin{bmatrix} R_1 & t_1 \\ 0^\top & 1 \end{bmatrix}^{-1} X_{\text{cam,1}} = X
\]

\[
x_2 = K_2[I | 0] \begin{bmatrix} R_2 & t_2 \\ 0^\top & 1 \end{bmatrix} X
\]

\[
x_2 = K_2[I | 0] \begin{bmatrix} R_2 & t_2 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} R_1 & t_1 \\ 0^\top & 1 \end{bmatrix}^{-1} X_{\text{cam,1}}
\]

\[
x_2 = K_2[I | 0] \begin{bmatrix} R_2 & t_2 \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} R_1^\top & -R_1^\top t_1 \\ 0^\top & 1 \end{bmatrix} X_{\text{cam,1}}
\]

\[
x_2 = K_2[I | 0] \begin{bmatrix} R_2 R_1^\top & t_2 - R_2 R_1^\top t_1 \\ 0^\top & 1 \end{bmatrix} X_{\text{cam,1}}
\]

\[
x_2 = K_2[I | 0] \begin{bmatrix} R & t \\ 0^\top & 1 \end{bmatrix} X_{\text{cam,1}}
\]

where \(R = R_2 R_1^\top\) and

\(t = t_2 - R_2 R_1^\top t_1\)
Image points

• Image points in pixel coordinates

\[ x = K[I | 0] \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} X \]

\[ x = K[R | t]X \]

• Image points in normalized coordinates

\[ x = K[R | t]X \]
\[ K^{-1}x = [R | t]X \]
\[ \hat{x} = [R | t]X \text{ where } \hat{x} = K^{-1}x \]
Image points in normalized coordinates

\[
x = K[I \mid 0] \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} X
\]

\[
K^{-1}x = [I \mid 0] \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} X
\]

\[
\hat{x} = [I \mid 0] \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} X \text{ where } \hat{x} = K^{-1}x
\]

\[
\hat{x} = [I \mid 0]X_{\text{cam}}
\]

\[
\hat{x} = [I \mid 0] \begin{bmatrix} \hat{X}_{\text{cam}} \\ 1 \end{bmatrix}
\]

\[
\hat{x} = \hat{X}_{\text{cam}} \text{ (up to nonzero scale)}
\]
The essential matrix

3D point $\tilde{X}' = \lambda'\hat{x}'$ in the second camera coordinate frame.
3D point $\tilde{X} = \lambda\hat{x}$ in the first camera coordinate frame.
Map $\tilde{X}$ from first camera coordinate frame to second camera coordinate frame.

\[
\tilde{X}' = R\tilde{X} + t
\]
\[
\tilde{X}' = R(\lambda\hat{x}) + t
\]
\[
\tilde{X}' = \lambda R\hat{x} + t
\]
The essential matrix

3D point $\tilde{\mathbf{X}}' = \lambda' \hat{\mathbf{x}}'$ in the second camera coordinate frame.
3D point $\tilde{\mathbf{X}} = \lambda \hat{\mathbf{x}}$ in the first camera coordinate frame.
Map $\tilde{\mathbf{X}}$ from first camera coordinate frame to second camera coordinate frame.

$$
\tilde{\mathbf{X}}' = \mathbf{R} \tilde{\mathbf{X}} + \mathbf{t}
$$

$\tilde{\mathbf{X}}' = \mathbf{R} (\lambda \hat{\mathbf{x}}) + \mathbf{t}$

$\tilde{\mathbf{X}}' = \lambda \mathbf{R} \hat{\mathbf{x}} + \mathbf{t}$

$$
\lambda \mathbf{R} \hat{\mathbf{x}} + \mathbf{t} = \lambda' \hat{\mathbf{x}}'
$$

$$
[t]_\times (\lambda \mathbf{R} \hat{\mathbf{x}} + \mathbf{t}) = [t]_\times (\lambda' \hat{\mathbf{x}}')
$$

$\lambda [t]_\times \mathbf{R} \hat{\mathbf{x}} = \lambda' [t]_\times \hat{\mathbf{x}}'$

$$
\hat{\mathbf{x}}'^\top (\lambda [t]_\times \mathbf{R} \hat{\mathbf{x}}) = \hat{\mathbf{x}}'^\top (\lambda' [t]_\times \hat{\mathbf{x}}')
$$

$$
\lambda \hat{\mathbf{x}}'^\top [t]_\times \mathbf{R} \hat{\mathbf{x}} = 0
$$

$$
\hat{\mathbf{x}}'^\top [t]_\times \mathbf{R} \hat{\mathbf{x}} = 0
$$

The epipolar constraint $\hat{\mathbf{x}}'^\top \mathbf{E} \hat{\mathbf{x}} = 0$ where $\mathbf{E} = [t]_\times \mathbf{R}$

Essential Matrix
(Longuet-Higgins, 1981)
Cross product using a skew symmetric matrix

- The cross product $\mathbf{a} \times \mathbf{b}$ of two 3-vectors $\mathbf{a} = (a_1, a_2, a_3)^T$ and $\mathbf{b} = (b_1, b_2, b_3)^T$ can be expressed as a matrix-vector product $[\mathbf{a}]_\times \mathbf{b}$, where $[\mathbf{a}]_\times$ is the 3x3 skew symmetric matrix

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- A matrix $\mathbf{S}$ is skew symmetric if and only if $\mathbf{S} = -\mathbf{S}^T$
- The determinant of a skew symmetric matrix is 0
The essential matrix

• Maps a point (in normalized coordinates) in the first image to its corresponding epipolar line (in normalized coordinates) in the second image

\[ \hat{\ell}' = E\hat{x} \]

• The epipolar line passes through the corresponding point in the second image

\[ \hat{x}'^\top \hat{\ell}' = 0 \]
\[ \hat{x}'^\top E\hat{x} = 0 \quad \text{The epipolar constraint} \]

• Every epipolar line passes through the epipole

\[ \hat{e}'^\top \hat{\ell}' = 0 \]
The essential matrix

• Maps a point (in normalized coordinates) in the second image to its corresponding epipolar line (in normalized coordinates) in the first image

\[ \hat{\ell} = E^T \hat{x}' \]

\[ \hat{\ell}^T = \hat{x}'^T E \]

• The epipolar line passes through the corresponding point in the first image

\[ \hat{\ell}^T \hat{x} = 0 \]

\[ \hat{x}'^T E \hat{x} = 0 \quad \text{The epipolar constraint} \]

• Every epipolar line passes through the epipole

\[ \hat{\ell}^T \hat{e} = 0 \]
Example of using the essential matrix
Another example: converging cameras

courtesy of Andrew Zisserman
Another example: second camera in front of first camera

courtesy of Andrew Zisserman
Epipolar rectify stereo images
• Epipolar geometry reduces matching complexity from $O(n^4)$ to $O(n^3)$
• But matching requires comparing points across pairs of epipolar lines which may have arbitrary orientation. That can be costly to index.
• Is there a more convenient epipolar geometry?

Slanted epipolar lines vs Horizontal, row aligned epipolar lines
Cameras with a convenient epipolar geometry

• When two cameras have parallel optical axes and these axes are orthogonal to the baseline, the epipolar lines are parallel.

• When rows of the two images are parallel to the baseline, the epipolar lines are horizontal rows of the two images.
Cameras with a convenient epipolar geometry

- When two cameras have parallel optical axes and these axes are orthogonal to the baseline, the epipolar lines are parallel.

- When rows of the two images are parallel to the baseline, the epipolar lines are horizontal rows of the two images.
What if stereo geometry is not convenient?

Rectification: Given a pair of images, transform both images so that epipolar lines are image rows.
Epipolar rectification

• Given a pair of images, transform both images so that epipolar lines are image rows
• Create pair of virtual cameras
  – The virtual cameras have the same camera centers as real cameras
  – Both virtual cameras have the same:
    • Camera rotation matrix R
    • Camera calibration matrix K
Epipolar rectification

• Given calibrated stereo cameras (i.e., $K_1$, $R_1$, $t_1$, $K_2$, $R_2$, $t_2$) determine the (same) rotation matrix $R$ and (same) calibration matrix $K$ of the virtual cameras

• To minimize image distortion:
  – For the calibration matrix
    • For principal point, set $x_0$ and $y_0$ to average of input $x_0$ and $y_0$ values, respectively
    • Set two focal length parameters to average of all four input focal length parameters (results in square pixels)
    • Set skew to 0
  – For the rotation matrix $R$, interpolate halfway between the two 3D rotations embodied by $R_1$ and $R_2$
Epipolar rectification

• For the rotation matrix $\mathbf{R}$, interpolate halfway between the two 3D rotations embodied by $\mathbf{R}_1$ and $\mathbf{R}_2$
  
  – Rotating a camera does not change its center $\mathbf{C}$, but does change its translation $\mathbf{t}$

\[
\tilde{\mathbf{o}}_{\text{cam}} = \mathbf{R}\tilde{\mathbf{C}} + \mathbf{t} \\
-\mathbf{R}\tilde{\mathbf{C}} = \mathbf{t} \\
\tilde{\mathbf{C}} = -\mathbf{R}^T\mathbf{t} \\
x = \mathbf{K}[\mathbf{R} \mid \mathbf{t}]\mathbf{X} \\
x = \mathbf{K}[\mathbf{R} \mid -\mathbf{R}\tilde{\mathbf{C}}]\mathbf{X} \\
x = \mathbf{KR}[\mathbf{I} \mid -\tilde{\mathbf{C}}]\mathbf{X}
\]
Rectification transformation matrices

• Transformation from image acquired by real camera to image acquired by virtual camera

\[
\begin{align*}
\mathbf{x}_{\text{real}} &= \mathbf{K}_{\text{real}} \mathbf{R}_{\text{real}} \begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix} \mathbf{X} \\
\mathbf{R}_{\text{real}}^{\top} \mathbf{K}_{\text{real}}^{-1} \mathbf{x}_{\text{real}} &= \begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix} \mathbf{X}
\end{align*}
\]

\[
\begin{align*}
\mathbf{x}_{\text{virtual}} &= \mathbf{K}_{\text{virtual}} \mathbf{R}_{\text{virtual}} \begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix} \mathbf{X} \\
\mathbf{x}_{\text{virtual}} &= \mathbf{K}_{\text{virtual}} \mathbf{R}_{\text{virtual}} \mathbf{R}_{\text{real}}^{\top} \mathbf{K}_{\text{real}}^{-1} \mathbf{x}_{\text{real}} \\
\mathbf{x}_{\text{virtual}} &= \mathbf{H} \mathbf{x}_{\text{real}}, \text{ where } \mathbf{H} = \mathbf{K}_{\text{virtual}} \mathbf{R}_{\text{virtual}} \mathbf{R}_{\text{real}}^{\top} \mathbf{K}_{\text{real}}^{-1}
\end{align*}
\]

\[
\begin{bmatrix}
\mathbf{x}_{\text{virtual}} \\
\mathbf{y}_{\text{virtual}} \\
\mathbf{w}_{\text{virtual}}
\end{bmatrix} =
\begin{bmatrix}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_{\text{real}} \\
\mathbf{y}_{\text{real}} \\
\mathbf{w}_{\text{real}}
\end{bmatrix}
\]
Rectification

Under perspective projection, the mapping from a plane to a plane is given by a 2D projective transformation (homography)
Rectification

Under perspective projection, the mapping from a plane to a plane is given by a 2D projective transformation (homography)

\[
\begin{bmatrix}
    x_L \\
    y_L \\
    w_L
\end{bmatrix}
= H_L
\begin{bmatrix}
    u_L \\
    v_L \\
    1
\end{bmatrix}
\]

Two homographies \( H_L, H_R \)

\[
\begin{bmatrix}
    x_R \\
    y_R \\
    w_R
\end{bmatrix}
= H_R
\begin{bmatrix}
    u_R \\
    v_R \\
    1
\end{bmatrix}
\]
Image pair rectification

Apply projective transformation so that epipolar lines correspond to horizontal scanlines

\[ H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

H should map epipole \( e \) to \((1,0,0)\), a point at infinity on the x-axis.

H should minimize image distortion.

Note that rectified images are usually not rectangular.
Rectification

Given a pair of images, transform both images so that epipolar lines are scan lines.

Input Images
Rectification

Given a pair of images, transform both images so that epipolar lines are scan lines.

Rectified Images

Epipolar lines run parallel with the x-axis and are aligned between two views (no y disparity)
Epipolar rectification

• Input: Source image I and Rectification matrix H

• Forward “warping” method
  – For each pixel coordinate $x_{\text{real}}$ in source image, map it to pixel coordinate $x_{\text{virtual}}$ in target image
    \[ x_{\text{virtual}} = H x_{\text{real}} \]
  – Problem: there is no guarantee that every pixel in target Image will be written to
    • If target image is larger than source image or target image is highly stretched, then there may be missing points that appear as speckles or lines
Epipolar rectification

- Input: Source image $I$ and Rectification matrix $H$
- Backward “warping” method
  - For each pixel coordinate $x_{\text{virtual}}$ in target image, map it to pixel coordinate $x_{\text{real}}$ in source image
    \[ x_{\text{real}} = H^{-1}x_{\text{virtual}} \]
  - Interpolate pixel values in source image to determine pixel value in destination image
  - Problem: some of the pixel coordinates in the target image map to pixel coordinates outside of the source image
    - You’ll solve this problem in a homework assignment by determining where the pixel coordinates of the corners of the source image map to pixel coordinates in the destination image using
    \[ x_{\text{virtual}} = Hx_{\text{real}} \]
Rectification

Original

Rectified
Rectification

- Epipolar lines
Rectification

Original

Rectified
Polar Rectification

Homography-based Rectification

Polar Rectification

Alternative epipolar rectification method that minimizes pixel distortion
Polar Rectification

Epipoles are in images (white dot on ball)

Homography-based rectification is not possible
Features on same epipolar line
Next Lecture

• Calibrated stereo and feature matching