CSE248: ALGORITHMIC AND OPTIMIZATION FOUNDATIONS FOR VLSI CAD

Partitioning

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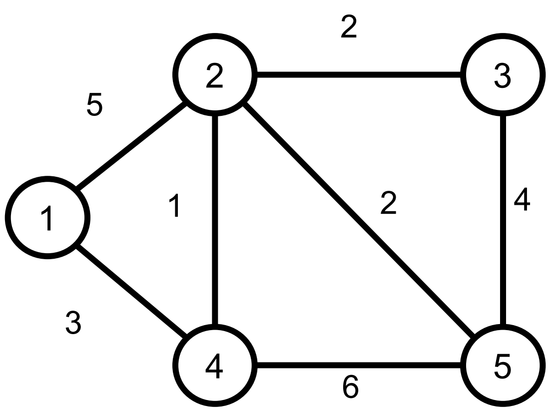
Outlines

- Min Cuts
- Gomory-Hu Cut Tree
- Ancestor Tree
- Replication Cut
- Partitioning with Retiming

Min Cuts

- Maximum flow minimum cut: Given a graph $G(V,E)$, and a pair of nodes s, and t, the maximum flow from s to t forms a minimum cut. $\min C(X,\bar{X})$, where $s \in X, t \in \bar{X}, X \cup \bar{X} = V$
- Ratio Cut: $\min R(X,\bar{X}) = C(X,\bar{X})/(|X||\bar{X}|)$



A Network Example

Maximum Flow Minimum Cuts

Given a directed graph $G(V, E)$, with $s, t \in V$.

$\max f$

$\sum_{ij\in E} x_{ij}$-$\sum_{ji\in E} x_{ji} = f \; if \; i = s,$

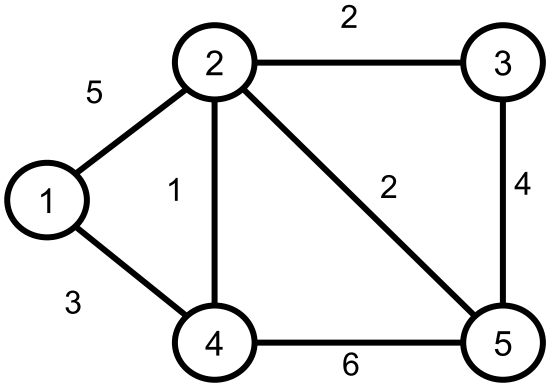
$\sum_{ij\in E} x_{ij}$-$\sum_{ji\in E} x_{ji} = 0 \; if \; i \in V - \{s, t\},$

$\sum_{ij\in E} x_{ij}$-$\sum_{ji\in E} x_{ji} = -f \; if \; i = t,$

$0 \leq x_{ij} \leq c_{ij}$ for all $ij \in E$

Min Cuts

- Maximum flow minimum cut: Given a graph $G(V, E)$, and a pair of nodes s, and t, the maximum flow from s to t forms a minimum cut. $\min C(X, \bar{X})$, where $s \in X, t \in \bar{X}, X \cup \bar{X} = V$
  - Maximum flow from node s to node t formulation
  - Dual of the maximum flow formulation
- Ratio Cut: $\min R(X, \bar{X}) = C(X, \bar{X}) / (|X||\bar{X}|)$
  - Multicommodity flow
  - Dual of the multicommodity flow

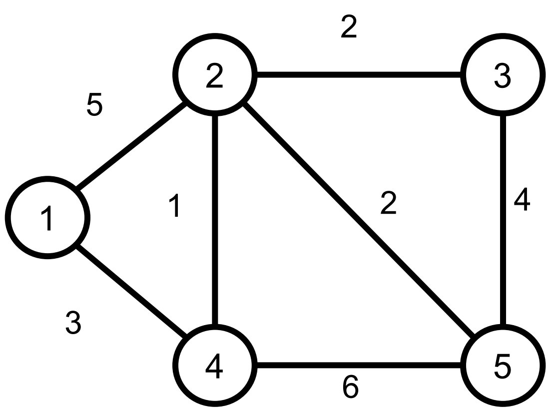


Cut Trees (undirected graph)

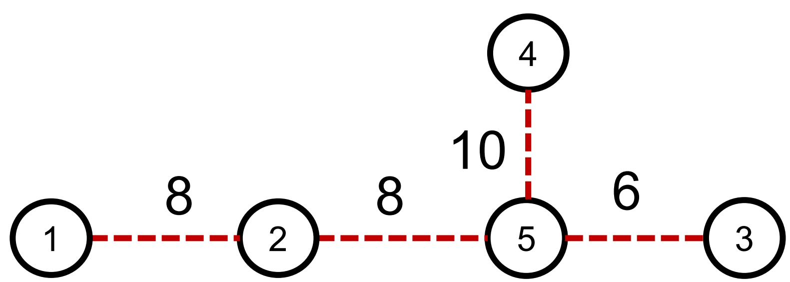
- Gomory and Hu's Cut Tree
  - Tree representation of all pairs of maximum flor minimum cuts
  - Journal of SIAM, 1961
- Ancestor Tree
  - Tree representation of all pairs of cuts (arbitrary objective function)
  - Annals of Operations Research ,1991

The Gomory-Hu Cut Tree

- Maximum flow minimum cut: Given a graph, and a pair of nodes s, and t, the maximum flow from s to t forms a minimum cut.
- # pair of nodes: Given an undirected graph with n nodes, we can choose C(n, 2) pairs of nodes.
- Gomory and Hu: The $n-1$ minimum cuts determine the maximum flow between all pairs of nodes.



A Network Example



The Gomory-Hu Cut Tree

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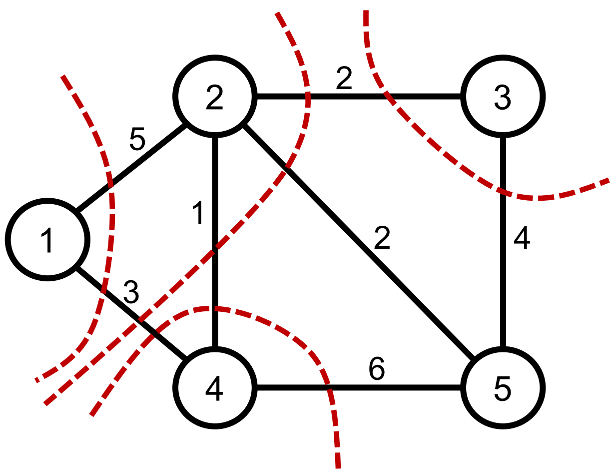
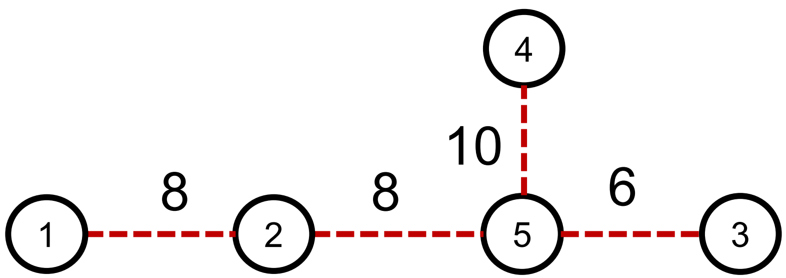


A Network Example

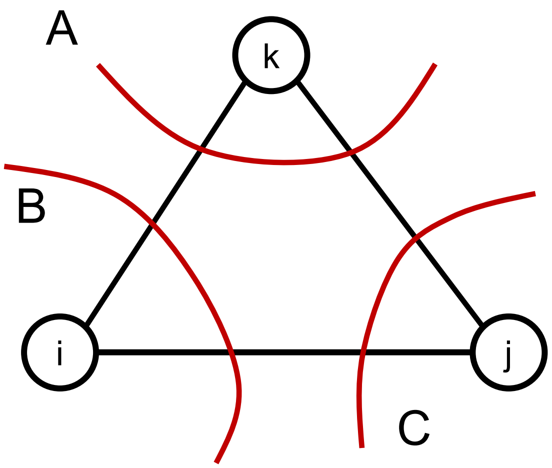


The Gomory-Hu Cut Tree

The Foundation of Cut Tree

- **Theorem**: A necessary and sufficient condition for a set of non-negative numbers $f_{ij} = f_{ji}\ (i, j = 1, \dots n)$ to be the minimum cut separating nodes i, j.

$$f_{ik} \geq min\left(f_{ij}, f_{jk}\right), \forall\ i, j, k$$



Lemma: For any three nodes of the network, at least two of the cut costs between them must be equal.

The Foundation of Cut Tree

- **By induction**, we have

$$f_{ip} \geq min\left(f_{ij}, f_{jk}, f_{kl}, \dots, f_{op}\right)$$

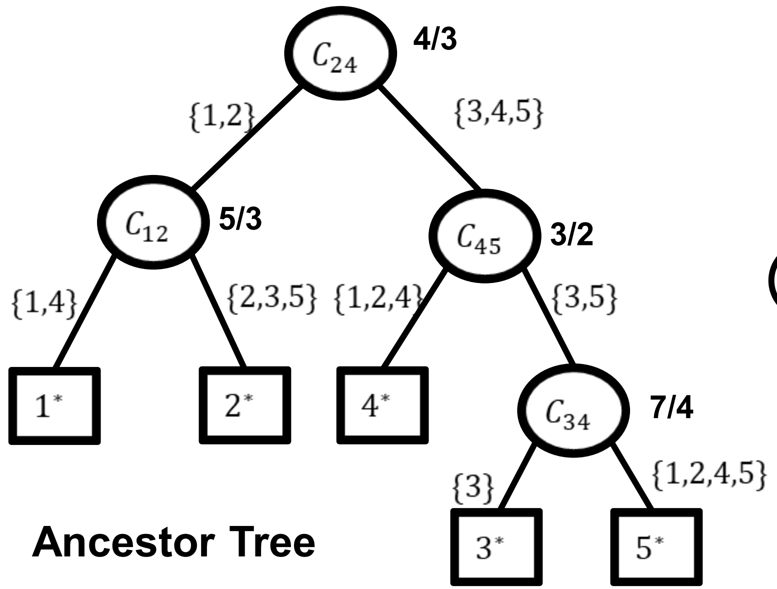
Where indices $i, j, \dots, p$ represent an arbitrary **sequences of nodes in the network**

Lemma: There is no loop in the cut representation.

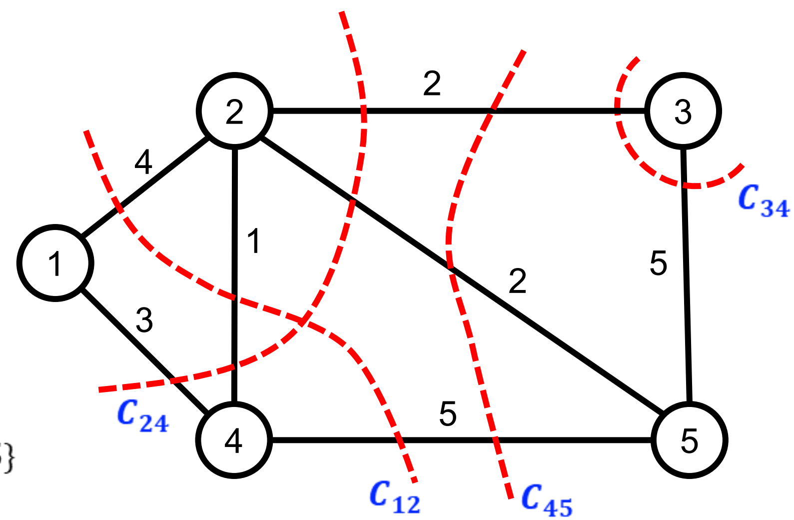
The Ancestor Tree

- The minimum cut tree for an arbitrary cut cost.

Ratio Cut Example: $F_{ij} = \min \frac{c(X,\bar{X})}{|X|\cdot|\bar{X}|}$ with nodes $i \in X$ and $j \in \bar{X}$. Ratio cut is an NP-complete problem.

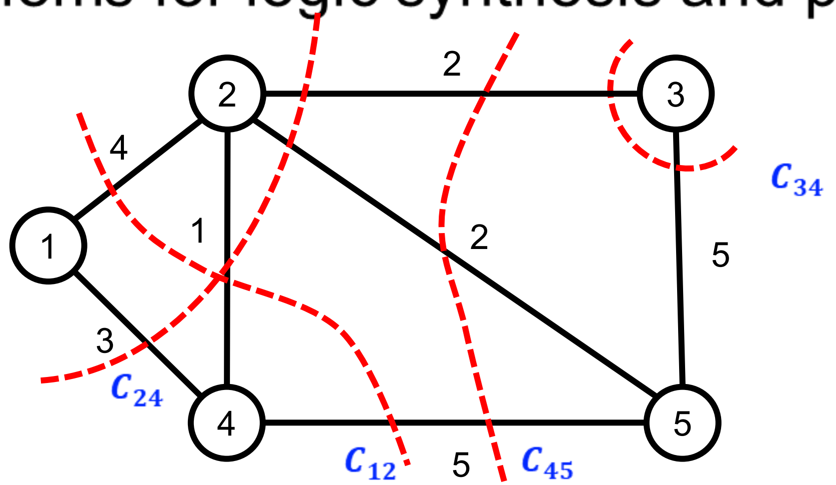


Ancestor Tree



The Properties of Ancestor Tree

- **The ancestor tree algorithms derives the essential cut set with $n-1$ minimum cut calls.**
- The technique has been applied to solving complex multi-commodity network optimization problems as well as network partitioning problems
- This partitioning can be further applied to solve VLSI design problems for logic synthesis and physical layout.



References of Ancestor Tree

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