CSE 207B: Applied Cryptography

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UCSD

Fall 2023 Lecture 11

Announcements

1. HW 5 is due in one week!
2. HW 6 is online!

Last time:

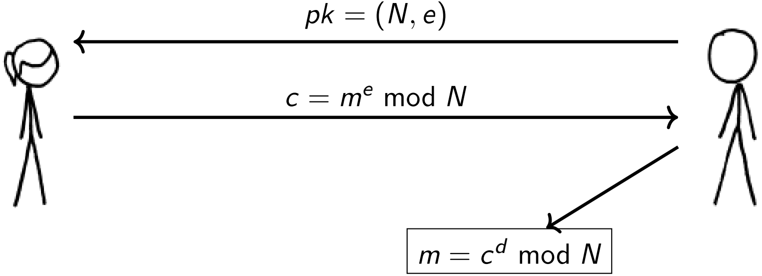
- RSA

This time:

- Attacks on RSA
- CCA security

Reminder: Textbook RSA Encryption

- Key Generation:
  1. $N = pq$
  2. Choose e s.t. $\gcd(e, \phi(N)) = 1$
  3. $d = e^{-1} \bmod \phi(N)$
  4. $pk = (N, e)$, $sk = (N, d)$.
- Encryption: $c = m^e \bmod N$
- Decryption: $m = c^d \bmod N$



RSA Key Generation Vulnerabilities

Common moduli, different exponents

If $pk_1 = (e_1, N)$ and $pk_2 = (e_2, N)$

Factorization of N reveals $d = e^{-1} \bmod (p-1)(q-1)$ for any e.

RSA Key Generation Vulnerabilities

Common moduli, different exponent and encryption

Let $pk_1 = (e_1, N)$ and $pk_2 = (e_2, N)$.

Encrypt the same m to both keys above:

$$c_1 = m^{e_1} \bmod N \qquad c_2 = m^{e_2} \bmod N$$

If $\gcd(e_1, e_2) = 1$ compute $ae_1 + be_2 = 1$

$$c_1^a c_2^b = m^{e_1 a} m^{e_2 b} = m \bmod N$$

RSA is homomorphic under multiplication

If we have a ciphertext $c = m^e \bmod N$, can forge encryption of mr by computing

$$cr^e \bmod N = m^e r^e \bmod N = (mr)^e \bmod N$$

Implications:

- Positive use: blinding. Can blind ciphertexts before decryption to try to prevent side-channel attacks, or blind signatures before signing. (More later.)
- Negative use: Chosen ciphertext attacks.

Chosen Ciphertext Attack Game for Public-Key Encryption

A $\qquad$ C

$\xleftarrow{pk}$ $\qquad$ pk, sk

Encryption queries:

$\xrightarrow{(m_{i0}, m_{i1})}$ $\qquad$ $b \in \{0,1\}$

$\xleftarrow{Enc_{pk}(m_{ib}) = c_i}$

Decryption queries:

$\hat{c}_j \notin \{c_1, \ldots, c_k\}$ $\xrightarrow{\hat{c}_j}$

$\xleftarrow{\hat{m}_j = Dec_{sk}(\hat{c}_j)}$

Definition

$(\mathrm{Enc}, \mathrm{Dec})$ is CCA-secure if

$|\Pr[A = 1 | b = 0] - \Pr[A = 1 | b = 1]|$ is negligible.

Chosen ciphertext attack on textbook RSA

1. Input challenge ciphertext $c = m^e \bmod N$.
2. Submit ciphertext $c' = r^e c \bmod N$ for decryption.
3. Receive message $m' = rm$.
4. Original message is $m' r^{-1} \bmod N = m$.

CCA-Secure RSA encryption

Our hybrid RSA encryption from last lecture is also CCA secure.

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  1. Generate primes p, q; $N = pq$
  2. Choose odd e s.t. $\gcd(e, \phi(N)) = 1$
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- Encryption: Choose random x, $y = x^e \bmod N$; $k = H(x)$;
  $c = \mathsf{SymEnc}_k(m)$. Send (y, c).

- Decryption: Input (y, c). $x = y^d \bmod N$; $k = H(x)$;
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Unfortunately, nobody actually uses this in practice.

RSA Padding Schemes

To protect against RSA malleability, RSA is universally used with a padding scheme in practice.

Instead of $\mathsf{Enc}_{pk}(m) = m^e \bmod N$, we define:

- $\mathsf{Enc}_{pk}(m) = (\mathsf{pad}(m))^e \bmod N$
- $\mathsf{Dec}_{sk}(m)$:
    1. Compute $p = c^d \bmod N$.
    2. If p has correct padding format, return $\mathsf{unpad}(p)$.
    3. Else return "failure".

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You have seen this result in problems before.

PKCS #1 v. 1.5 padding

PKCS #1 v. 1.5 padding is the most common padding scheme for RSA in practice.
Encryption:

```
m = 00 02 [random padding string] 00 [data]
```

Signatures:

```
m = 00 01 FF .. FF 00 [data]
```

To decrypt, implementation checks padding format:

- First two byes correct.
- Padding string contains no null bytes.
- Presence of null byte.
- `data` is typically symmetric key data.

Bleichenbacher PKCS #1 v. 1.5 chosen ciphertext attack

[Bleichenbacher 1998]

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Attack setup:

- Attacker has a valid ciphertext c which is an encryption of a 48-byte SSL "premaster secret".
- Victim is a SSL 3.0 server with the private key.

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- Victim is a SSL 3.0 server with the private key.

1. Attacker queries server with candidates $cr^e \bmod N$.
2. $$\text{server} \begin{cases} \text{aborts if padding incorrect} \\ \text{continues if padding correct} \end{cases}$$
3. Server is padding oracle that leaks information about plaintext.

With a few million queries can decrypt a 2048-bit RSA ciphertext.

TLS countermeasures against Bleichenbacher attack

TLS 1.0–1.2 countermeasure:

- If padding incorrect, server generates fake plaintext and continues connection with that fake plaintext.
- Since client doesn't know secret, connection will fail later.

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A: Fears about backwards compatibility.

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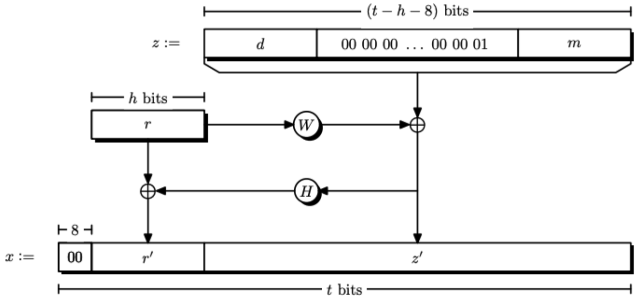
2016: DROWN Attack

- Since servers use the same RSA keys with old versions of SSL/TLS, attacker can mount Bleichenbacher attack against servers supporting SSL 2.0 to decrypt a TLS ciphertext.

TLS 1.3 countermeasure: Eliminate RSA key exchange entirely.

OAEP: CCA-secure RSA padding

[Bellare Rogaway 1994], [Fujisaki et al.]



Uses hash functions H, W, optional associated data d.

Theorem

OAEP padding is CCA-secure in the random oracle model assuming that RSA is "partially one-way".

TLS, SSH, IPsec, etc. all default to PKCS#1 v. 1.5 padding.

Elementary factoring algorithms: Trial division

Input: $N \in \mathbb{Z}$
Output: p, $q \in \mathbb{Z}$ s.t. $pq = N$

Trial division:
For $i \leq \sqrt{N}$ check if $i \mid N$.

Elementary factoring algorithms: Pollard rho

Input: $N \in \mathbb{Z}$
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Pollard rho:
Take a random walk mod N, hope to find a cycle modulo $p \mid N$.

Problem: Want a collision modulo p, but we don't know p!
Solution: $a_i \equiv a_j \bmod p \implies p \mid \gcd(a_i - a_j, N)$

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Try #1: Generate $\sqrt{p} = O(N^{1/4})$ elements a_i.
Check $\gcd(a_i - a_j, N)$. Problem: $O(\sqrt{N})$ time.

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Try #2: Pseudorandom walk.
Define $f(x) = x^2 + c \bmod N$, our pseudorandom function.

1. Choose random starting point s, constant c. $a_1 = a_2 = s$
2. Iterate walk: $a_1 = f(a_1)$, $a_2 = f(f(a_2))$, compute $g = \gcd(a_1 - a_2, N)$.
   If $g = N$ start over. If $g \neq 1$ return g.

If f is sufficiently random, expect collision after $O(\sqrt{p})$ steps. N must have a factor p of size at most $O(\sqrt{N})$.

Elementary factoring algorithms: Pollard $p-1$

Input: $N \in \mathbb{Z}$
Output: $p,\ q \in \mathbb{Z}$ s.t. $pq = N$

Recall Fermat's little theorem: $a^{p-1} \equiv 1 \bmod p$.

1. Choose random a.
2. Compute $M(k) = \mathrm{lcm}(1 \ldots k) = \prod_i p_i^{e_i}, \qquad p_i^{e_i} < k$
3. Compute $b = a^{M(k)} - 1 \bmod N$.
4. Compute $\gcd(b, N) = g$.
5. If $g \neq 1$ or N return g.

Factors N if $p - 1 \mid M(k) \implies p-1$ has all small factors.

Countermeasure: Choose p so that $p-1$ has some big prime factors.

Advanced factoring algorithms: Number field sieve

Running time: $O(\exp(c(\lg N)^{1/3}(\lg \lg N)^{2/3}))$

Current record: RSA-250, 829 bits (February 2020)

RSA and GCDs

Public Key
$(N = pq, e)$

Private Key
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$$N_1 = pq_1 \qquad N_2 = pq_2$$

$$\gcd(N_1, N_2) = p$$

You can factor both keys with GCD algorithm.

Time to factor
829-bit RSA modulus:
2700 core-years
[Boudot et al. 2020]

Time to calculate GCD
for 1024-bit RSA moduli:
15μs

Naively computing pairwise GCDs

Euclid's algorithm $\gcd(a, b)$

if $b = 0$:

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else:

return $\gcd(b, a \bmod b)$

a, b have n bits $\rightarrow$ $O(n^2)$ time.

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for all pairs (N_i, N_j):

    if $\gcd(N_i, N_j) \neq 1$:

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$$\approx 1100 \text{ years}$$

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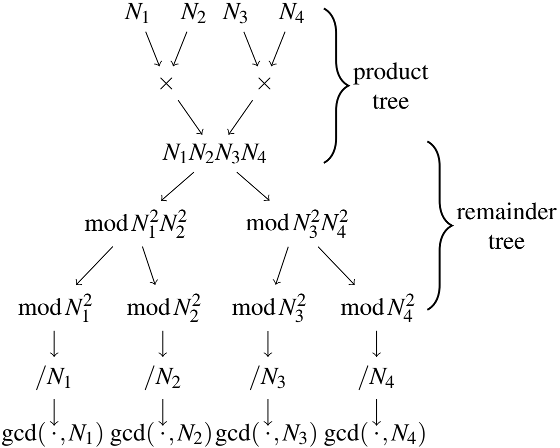
Efficiently computing pairwise GCDs

An efficient algorithm due to [Bernstein 2004].



$O(mn \operatorname{polylog}(mn))$ time for m n-bit integers, a few hours for internet-wide scan data.

Should we expect to find prime collisions in the wild?

Experiment: Compute GCD of each pair of M RSA moduli randomly chosen from P primes.

What *should* happen? **Nothing.**

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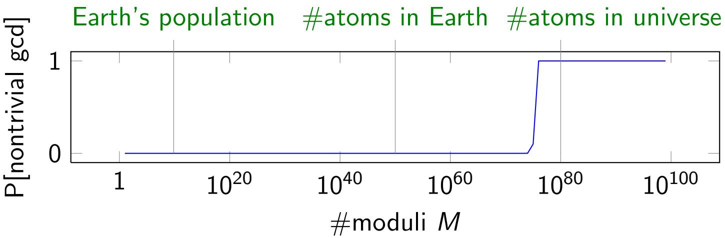
Prime Number Theorem:
$\sim 10^{150}$ 512-bit primes

Birthday bound:
$\Pr[\text{nontrivial gcd}] \approx 1 - e^{-2M^2/P}$



What happened when we GCDed RSA keys in 2012?

Computed private keys for

- 64,081 HTTPS servers (0.50%).
- 2,459 SSH servers (0.03%).
- 2 PGP users (and a few hundred invalid keys).

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What has happened since?

- 103 Taiwanese citizen smart card keys [Bernstein, Chang, Cheng, Chou, Heninger, Lange, van Someren 2013]
- 90 export-grade HTTPS keys. [Albrecht, Papini, Paterson, Villanueva-Polanco 2015]
- 313,330 HTTPS, SSH, IMAPS, POP3S, SMTPS keys [Hastings Fried Heninger 2016]
- 3,337 Tor relay RSA keys. [Kadianakis, Roberts, Roberts, Winter 2017]

Widespread RNG failures on low resource devices

We accidentally found *multiple independent cascading PRNG failures*.

Factor #1: Weak keys generated by low resource devices ($>$ 50 manufacturers).

1. Linux PRNG inputs: keyboard, mouse, disk
2. OpenSSL inputs: time, pid, OS PRNG
3. Headless or embedded devices lack these entropy sources.

Factor #2: Boot-time entropy hole on Linux PRNG

- Devices automatically generated keys on first boot.
- Linux PRNG had not yet been seeded when queried by OpenSSL.
- Fixed since July 2012.

2022 Linux PRNG Updates

"Random number generator enhancements for Linux 5.17 and 5.18"
`https://www.zx2c4.com/projects/linux-rng-5.17-5.18/`

- "the RNG can seed itself using cycle counter jitter in a second or so if it hasn't already been seeded by other entropy sources"
- "apparently we cannot yet unify /dev/random and /dev/urandom, because the day after this change made it to mainline breakage was detected on arm, m68k, microblaze, sparc32, and xtensa"
- "swapping out SHA-1 for BLAKE2s"
- "is 'premature next' a real world rng concern, or just an academic exercise?" `https://lore.kernel.org/lkml/YmlMGx6+uigkGiZ0@zx2c4.com/`

- Widespread RSA key generation and random number generation vulnerabilities were hiding in plain sight for years.
- Patching rates are low to nonexistent for networked devices.
- Gaps between theory and practice.