The Continuous Fourier Transform

Image Processing

CSE 166

Lecture 5

Announcements

- Assignment 2 is due Oct 18, 11:59 PM
- Assignment 3 will be released Oct 23
  - Due Oct 30, 11:59 PM
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.1 and 4.2

Overview: Image processing in the frequency domain

Image in spatial domain $f(x,y)$ → **Fourier transform** → Image in frequency domain $F(u,v)$

↓

Frequency domain processing

↓

Image in spatial domain $g(x,y)$ ← **Inverse Fourier transform** ← Image in frequency domain $G(u,v)$

Jean-Baptiste Joseph Fourier
1768-1830

Review

- Complex numbers
- Complex functions

Complex numbers

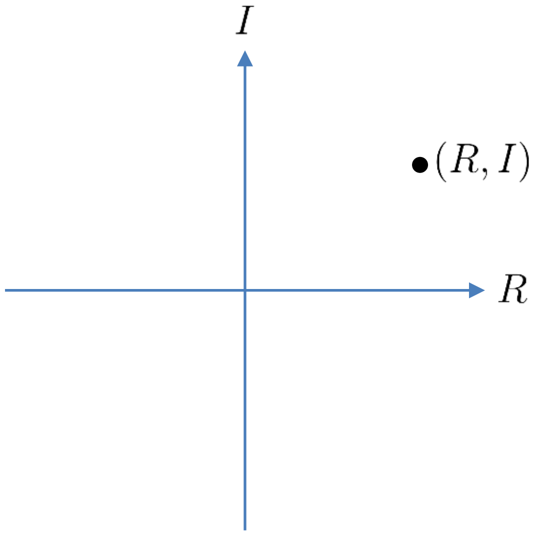
$$C = R + jI, \text{ where } j = \sqrt{-1}$$

Real numbers are a subset of complex numbers where $I = 0$

Complex conjugate

$$C^* = R - jI$$

Complex plane



Complex numbers

Magnitude (or complex modulus)

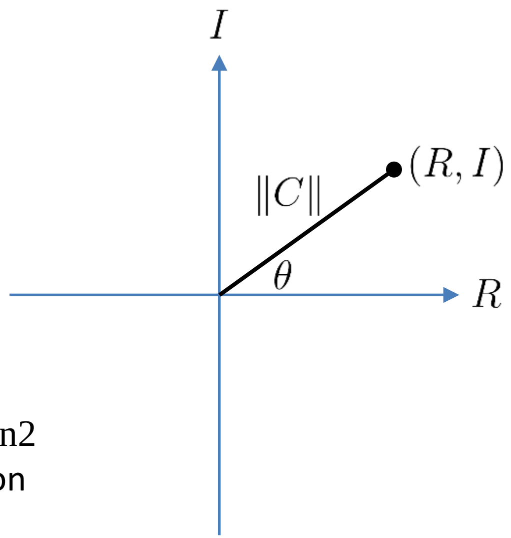
$$\|C\| = \sqrt{R^2 + I^2}$$

Angle (or phase or complex argument)

$$\tan(\theta) = \frac{I}{R} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\theta = \tan^{-1}\left(\frac{I}{R}\right)$$

Use atan2 function



Complex plane

Use these to convert from C to $\|C\|$ and θ

$$C = \|C\|(\cos(\theta) + j\sin(\theta))$$

$$C = \|C\|e^{j\theta}, \text{ where } e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Use this to convert from $\|C\|$ and θ to C

Euler's formula

Complex functions

$$F(u) = R(u) + jI(u)$$

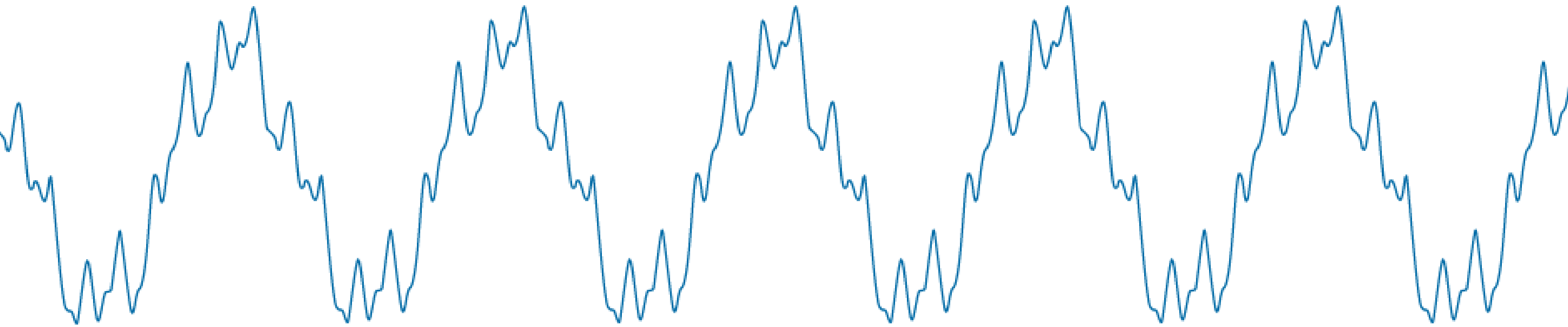
$$F^*(u) = R(u) - jI(u)$$ Complex conjugate

$$\|F(u)\| = \sqrt{R(u)^2 + I(u)^2}$$ Magnitude

$$\theta(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right)$$ Use atan2 function — Angle (or phase)

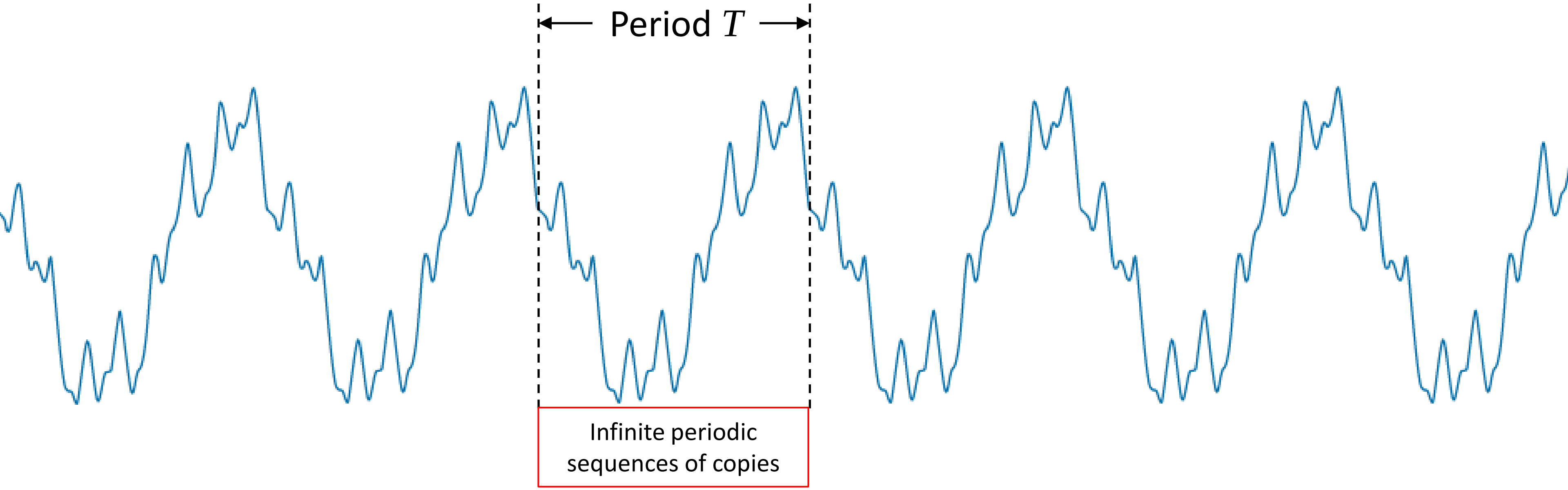
Periodic functions

- Example periodic function



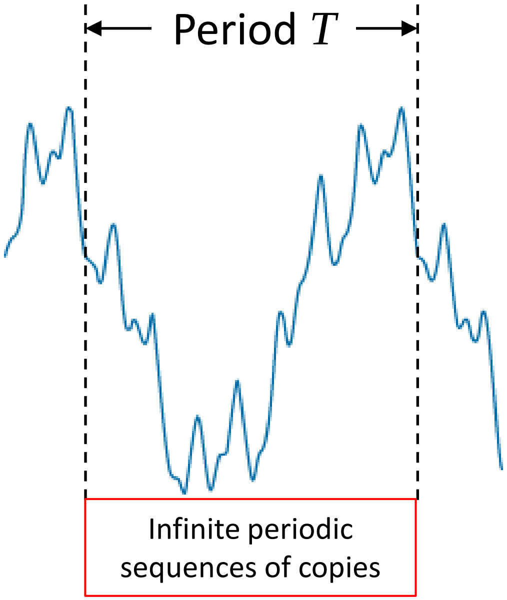
Periodic functions

- Example periodic function



Periodic functions

- Example periodic function



1D Fourier series

A function $f(t)$ of a continuous variable t that is periodic with period T can be expressed as a sum of sines and cosines multiplied by appropriate coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

where

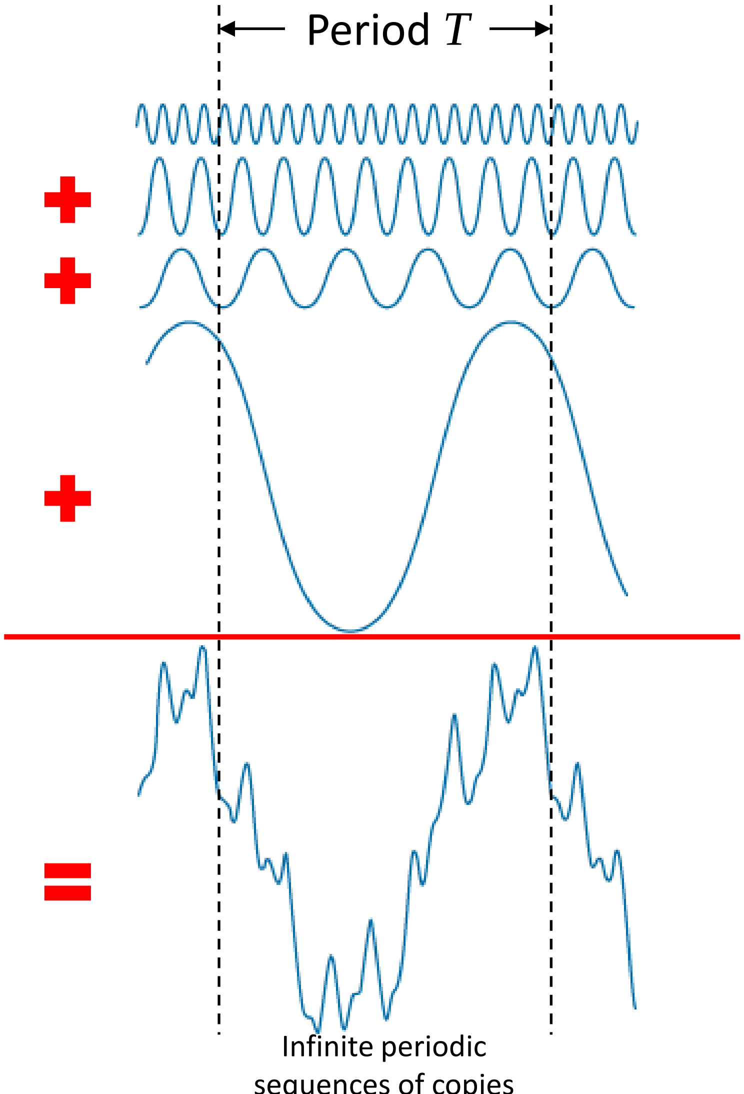
$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\frac{2\pi n}{T}t}\,\mathrm{d}t \text{ are the coefficients}$$

for $n = \ldots, -2, -1, 0, 1, 2, \ldots$

1D Fourier series

Sines and cosines

Periodic function



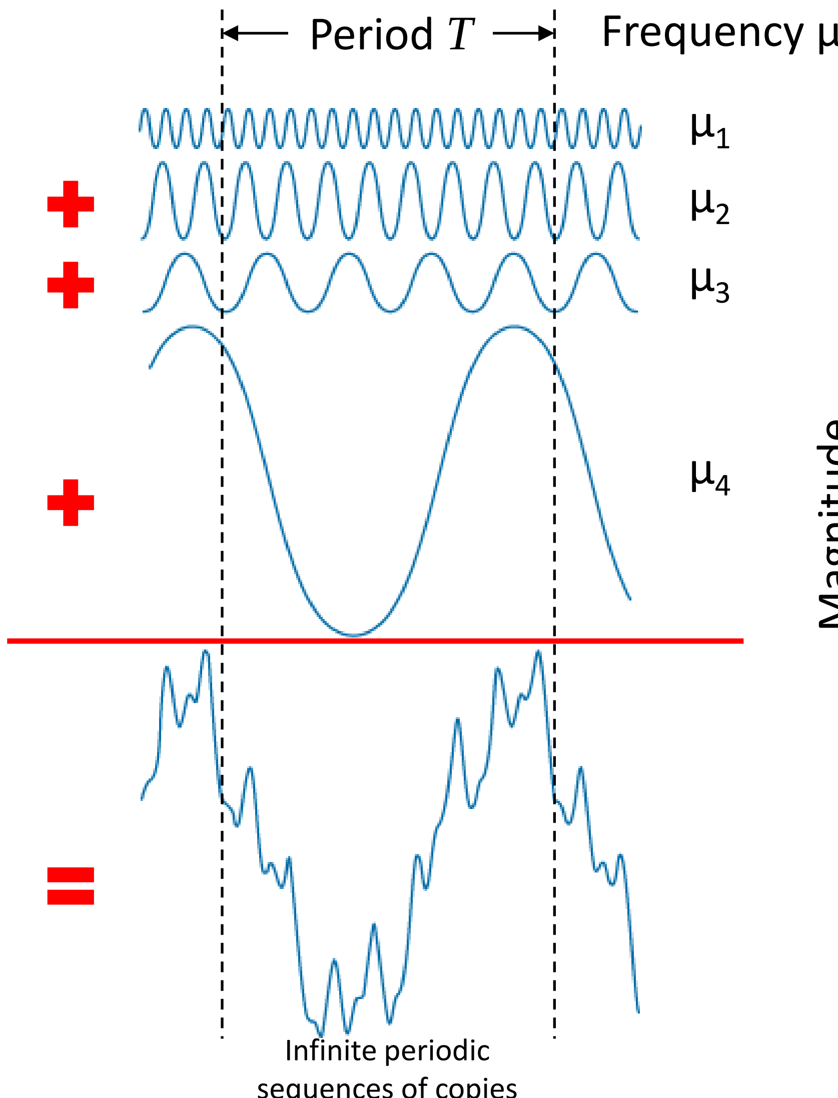
Weighted by magnitude

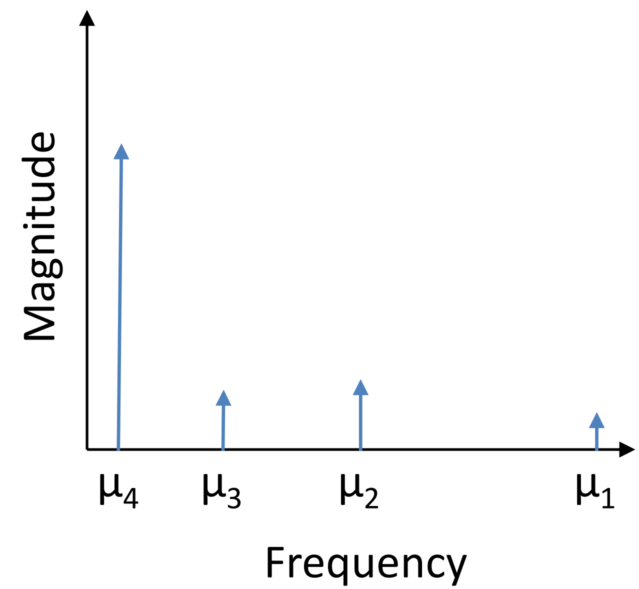
Shifted by phase

1D Fourier transform

Sines and cosines

Periodic function





1D continuous Fourier transform

- (Forward) Fourier transform

The continuous Fourier transform of a continuous function $f(t)$ is

$$\mathfrak{F}\left\{f(t)\right\} = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, \mathrm{d}t \text{ where } \mu \text{ is a continuous variable}$$

Since t is integrated out, $\mathfrak{F}\left\{f(t)\right\}$ is only a function of μ.

- Inverse Fourier transform

The inverse Fourer transform

$$\mathfrak{F}^{-1}\left\{F(\mu)\right\} = f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\, \mathrm{d}\mu \text{ where } t \text{ is a continuous variable}$$

μ is integrated out, so $\mathfrak{F}^{-1}\left\{f(\mu)\right\}$ is only a function of t.

1D continuous Fourier transform

- Fourier transform pair

$$\mathfrak{F}\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}^{-1}\{F(\mu)\} = f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\,\mathrm{d}\mu$$

1D continuous Fourier transform

- Using Euler's formula

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, \mathrm{d}t$$

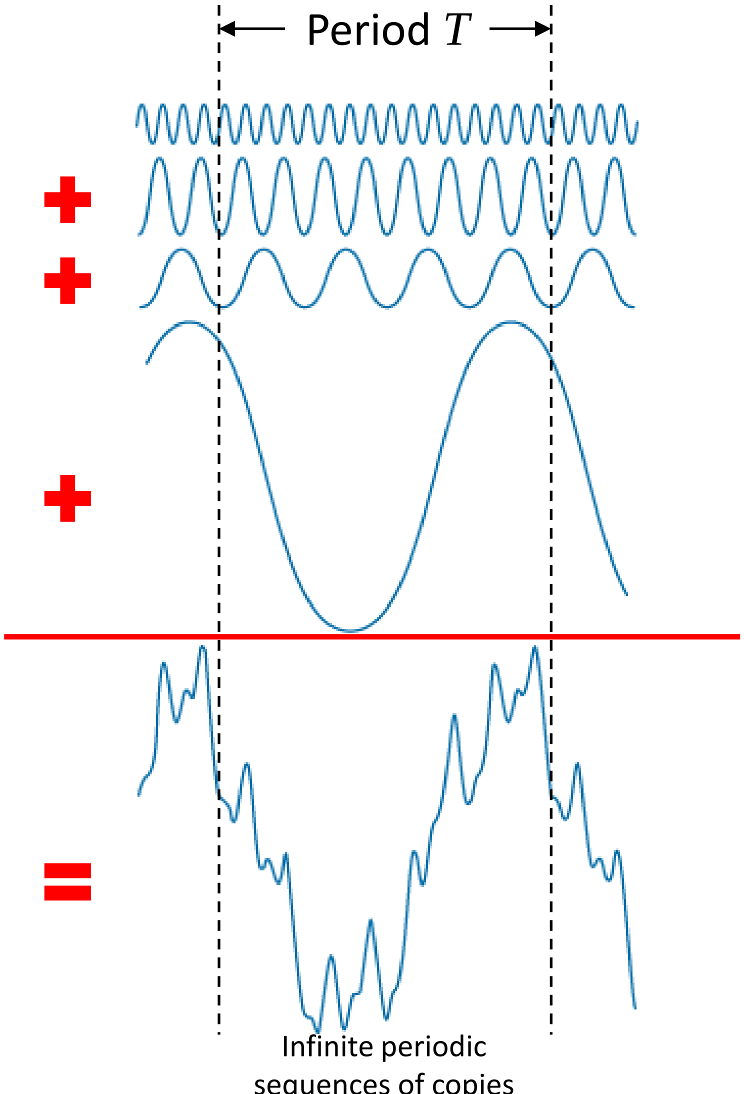
$$F(\mu) = \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi\mu t) - j\sin(2\pi\mu t)\right]\, \mathrm{d}t$$

Fourier transform is in frequency domain
Frequency μ is in units of cycles per unit of intput variable (e.g., cycles/seconds, cycles/meters).
If $f(t)$ is real, then (in general) $F(\mu)$ is complex. As such, it is customary to display the magnitude of $F(\mu)$, which is real.

1D Fourier series

Sines and cosines

Periodic function



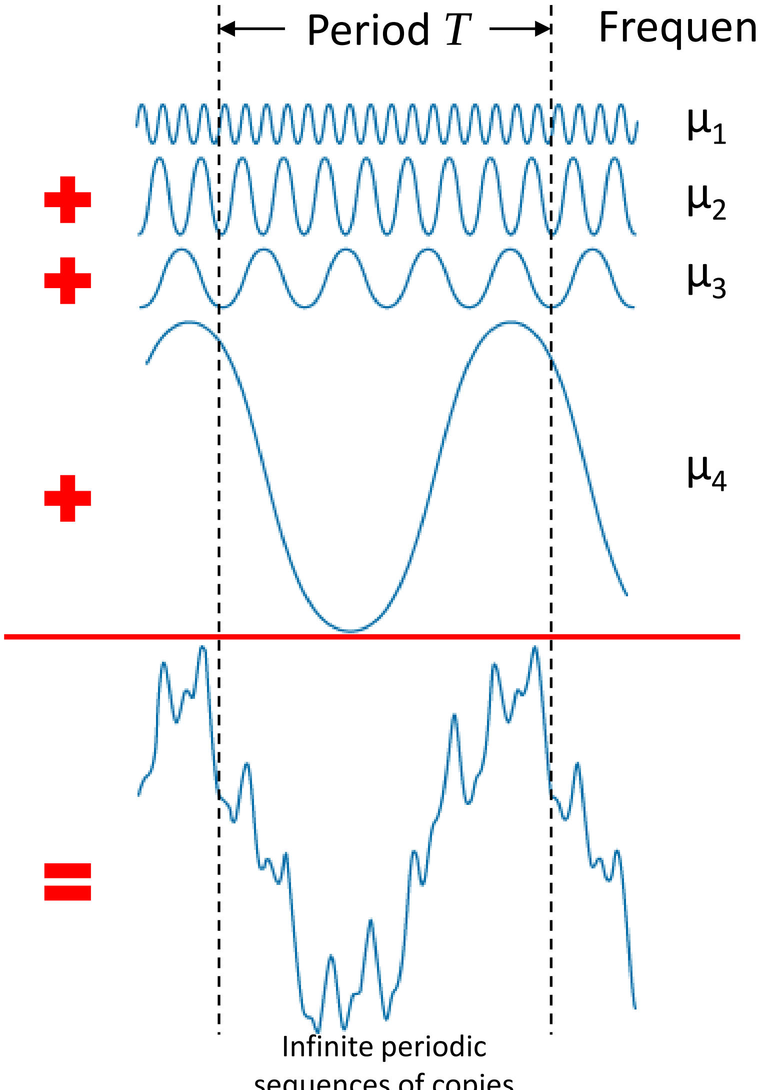
Weighted by magnitude

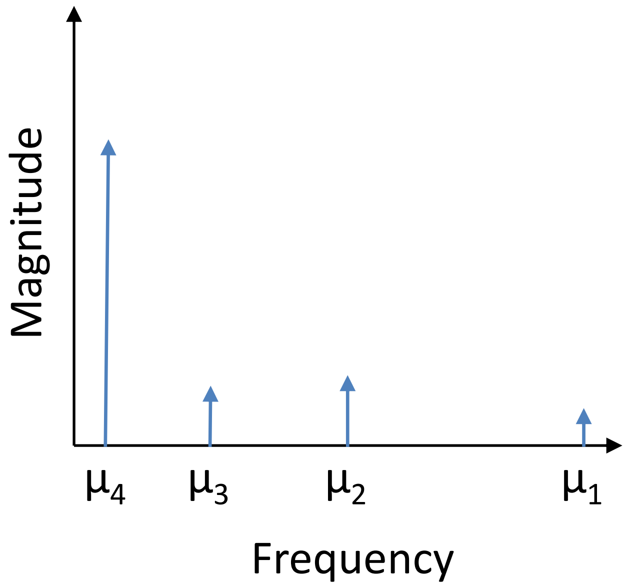
Shifted by phase

1D Fourier transform

Sines and cosines

Periodic function





1D continuous Fourier transform

- Example: box function
  - The Fourier transform of a box function does not have an imaginary component (i.e., it is real)

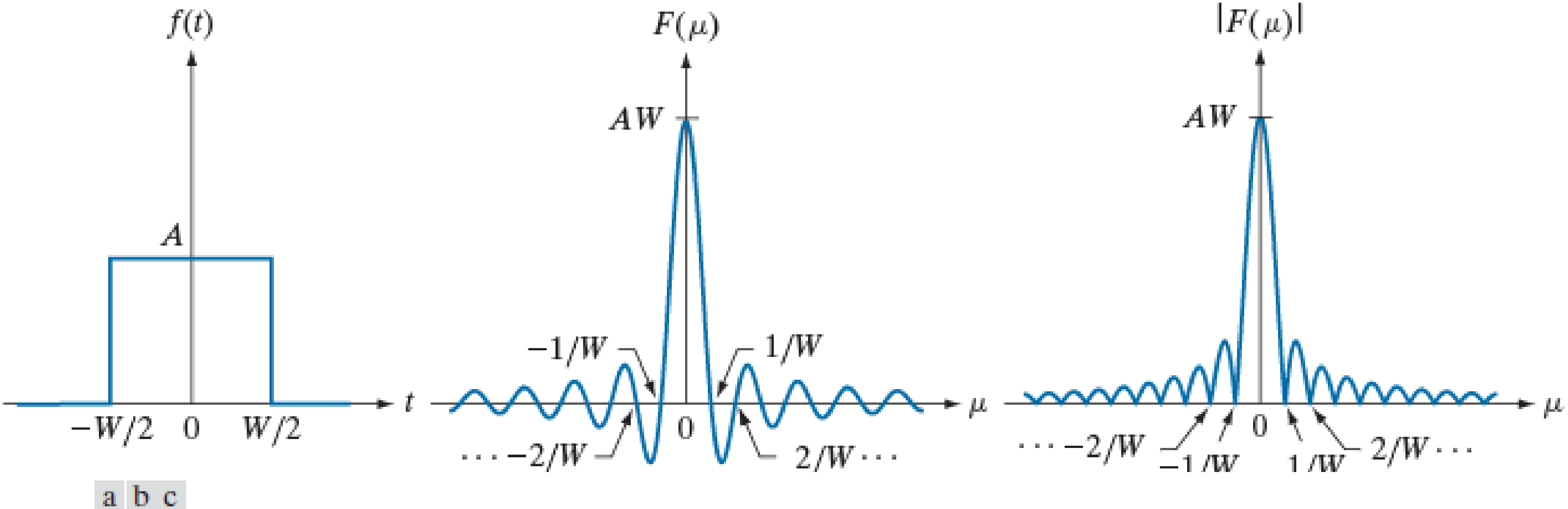


FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

1D continuous Fourier transform

- 1D continuous convolution

$$f(t)\bigstar h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,\mathrm{d}\tau\right] e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,\mathrm{d}t\right]\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,\mathrm{d}\tau$$

$$\mathfrak{F}\{f(t)\bigstar h(t)\} = H(\mu)F(\mu)$$

where

$$\mathfrak{F}\{h(t-\tau)\} = \int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}\,\mathrm{d}t$$

$$\mathfrak{F}\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau}$$

(Shown in future lecture)

1D continuous Fourier transform

- 1D convolution theorem

$$f(t)\bigstar h(t) \Longleftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t) \Longleftrightarrow H(\mu)\bigstar F(\mu)$$

Next Lecture

- Sampling and aliasing, and the discrete Fourier transform
- Reading
  - Chapter 4: Filtering in the Frequency Domain
    - Sections 4.2, 4.3, and 4.4