Geometric Image Formation

Computer Vision I
CSE 252A
Lecture 2
Announcements

• Assignment 0 will be released today
  – Due Oct 6, 11:59 PM
• Reading
  – Szeliski
    • Section 2.1
Earliest Surviving Photograph

• First photograph on record, “la table service” by Nicephore Niepce in 1822
• Note: First photograph by Niepce was in 1816
How Cameras Produce Images

• Basic process:
  – photons hit a detector
  – the detector becomes charged
  – the charge is read out as brightness

• Sensor types:
  – CCD (charge-coupled device)
    • high sensitivity
    • high power
    • cannot be individually addressed
    • blooming
  – CMOS
    • simple to fabricate (cheap)
    • lower sensitivity, lower power
    • can be individually addressed
Images are two-dimensional patterns of brightness values.

They are formed by the projection of 3D objects.
Lighting Affects Appearance: Monet
Viewpoint Affects Appearance: Monet

Haystack at Chailly at sunrise (1865)
Image Formation: Outline

• Geometric image formation
• Photometric image formation
• Color
Pinhole Camera: Perspective projection

- Abstract camera model - box with a small hole in it
Camera Obscura

"When images of illuminated objects ... penetrate through a small hole into a very dark room ... you will see [on the opposite wall] these objects in their proper form and color, reduced in size ... in a reversed position, owing to the intersection of the rays". --- Leonardo Da Vinci

http://www.acmi.net.au/AIC/CAMERA_OBSCURA.html (Russell Naughton)
Camera Obscura

• Used to observe eclipses (e.g., Bacon, 1214-1294)
• By artists (e.g., Vermeer).
Camera Obscura

Jetty at Margate England, 1898.

http://brightbytes.com/cosite/collection2.html (Jack and Beverly Wilgus)
A and C are same size, but A is further from camera, so its image A’ is smaller.
The projection of the point $P$ on the image plane $\Pi'$ is given by the point of intersection $P'$ of the ray defined by $PO$ with the plane $\Pi'$.
Equation of Perspective Projection

Cartesian coordinates:

- We have, by similar triangles, that for \( P = (x, y, z) \), the intersection of \( OP \) with \( \Pi' \) is \( (f' x/z, f' y/z, f') \)
- Establishing an image plane coordinate system at \( C' \) aligned with \( i \) and \( j \), we get \( (x, y, z) \rightarrow (f' \frac{x}{z}, f' \frac{y}{z}) \)
• Virtual image plane in front of optical center.
• Image is ‘upright’

\[(x, y, z) \rightarrow (f \frac{x}{z}, f \frac{y}{z})\]
Virtual Image Plane

- Virtual image plane in front of optical center.
- Image is ‘upright’

\[(x, y, z) \rightarrow (f \frac{x}{z}, f \frac{y}{z})\]
A Digression

Projective Geometry
and
Homogenous Coordinates
What is the intersection of two lines in a plane?

A Point
Do two lines in the plane always intersect at a point?

No, Parallel lines don’t meet at a point.
Can the perspective projection of parallel lines in 3D meet at a point in an image?

YES
Projective geometry provides an elegant means for handling these different situations in a unified way.

Homogeneous coordinates are a way to represent entities (points & lines) in projective spaces.
Homogeneous coordinates

• Boardwork
  – 2D points and lines
  – Point at infinity
  – Line at infinity
Homogeneous coordinates

• 3D point using inhomogeneous coordinates as 3-vector

\[
\tilde{X} = \begin{bmatrix}
\tilde{X} \\
\tilde{Y} \\
\tilde{Z}
\end{bmatrix}
\]

• 3D point using affine homogeneous coordinates as 4-vector

\[
X = \begin{bmatrix}
\tilde{X} \\
\tilde{Y} \\
\tilde{Z} \\
1
\end{bmatrix}
\]
Homogeneous coordinates

- 3D point using *affine* homogeneous coordinates as 4-vector
  \[ X = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \\ 1 \end{bmatrix} \]

- 3D point using *projective* homogeneous coordinates as 4-vector (*up to scale*)
  \[ X = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} \]
Homogeneous coordinates

- Projective homogeneous 3D point to affine homogeneous 3D point

\[
X = \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \frac{1}{W} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \\ 1 \end{bmatrix}
\]

- Dehomogenize 3D point

\[
\tilde{X} = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \end{bmatrix}
\]
Homogeneous coordinates

• Homogeneous points are defined up to a nonzero scale

\[
\begin{align*}
X &= \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \lambda \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} = \begin{bmatrix} \lambda X \\ \lambda Y \\ \lambda Z \\ \lambda W \end{bmatrix} \\
\tilde{X} &= \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \end{bmatrix} = \begin{bmatrix} \frac{X}{W} \\ \frac{Y}{W} \\ \frac{Z}{W} \end{bmatrix}
\end{align*}
\]
Homogeneous coordinates

• When $W = 0$, then it is a point at infinity
• Affine homogeneous coordinates are projective homogeneous coordinates where $W = 1$
• When not differentiating between affine homogeneous coordinates and projective homogeneous coordinates, simply call them homogeneous coordinates
End of the Digression
In a perspective image, parallel lines meet at a point, called the vanishing point. Doesn’t need to be near the center of the image.
Parallel lines meet in the image

- A single line $L$ can have a vanishing point.
- Vanishing point location: Intersection of image plane with a 3-D line $L^*$ through optical center $O$ parallel to $L$
Vanishing points

- A scene can have more than one vanishing point
- Different 3-D directions correspond different vanishing points
Vanishing Points
Vanishing Point

• In the **projective plane**, parallel lines meet at a point at infinity

• The 2D vanishing point in the image is the perspective projection of this 3D point at infinity
What is a Camera?

- A mathematical expression that relates points in 3D to points in an image for different types of physical cameras or imaging situations
Geometry

• How do 3D world points project to 2D image points?
The equation of projection

Inhomogeneous coordinates

\[(\tilde{X}, \tilde{Y}, \tilde{Z}) \rightarrow \left( f \frac{\tilde{X}}{\tilde{Z}}, f \frac{\tilde{Y}}{\tilde{Z}} \right)\]

Homogeneous coordinates

\[
\begin{bmatrix}
x \\
y \\
w \\
\end{bmatrix} =
\begin{bmatrix}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
W \\
\end{bmatrix}
\]

\[x = PX\]
What if camera coordinate system differs from world coordinate system?

Camera coordinate frame

World coordinate frame
Special cases

• Imaging a plane
• Only camera rotation (no translation)

• In both cases, mapping between images is a planar projective transformation

\[
\begin{bmatrix}
x' \\
y' \\
w'
\end{bmatrix} = \begin{bmatrix}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{bmatrix} \begin{bmatrix}
x \\
y \\
w
\end{bmatrix}
\]

\[x' = Hx\]
Imaging a plane

\[ x' = Hx \]

\[ x = PX_\pi \]

\[ x' = P'X_\pi \]
Only camera rotation (no translation)

\[ x = PX \]
\[ x' = P'X \]
\[ x' = Hx \]
Application: Panoramas and image stitching

All images are transformed to central image
Euclidean Coordinate Systems
Coordinate Change: Rotation Only

\[
\begin{bmatrix}
\tilde{X}' \\
\tilde{Y}' \\
\tilde{Z}'
\end{bmatrix} =
\begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix}
\begin{bmatrix}
\tilde{X} \\
\tilde{Y} \\
\tilde{Z}
\end{bmatrix}
\]

\[
\tilde{X}' = R\tilde{X}
\]

\[
\begin{bmatrix}
\tilde{X}' \\
1
\end{bmatrix} =
\begin{bmatrix}
R & 0 \\
0^\top & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{X} \\
1
\end{bmatrix}
\]

\[
X' =
\begin{bmatrix}
R \\
0^\top \\
0 \\
1
\end{bmatrix} X
\]
Some points about SO(n)

- $\text{SO}(n) = \{ R \in \mathbb{R}^{n\times n} : R^T R = I, \det(R) = 1 \}$
  - $\text{SO}(2)$: rotation matrices in plane $\mathbb{R}^2$
  - $\text{SO}(3)$: rotation matrices in 3-space $\mathbb{R}^3$

- Forms a Group under matrix product operation:
  - Identity
  - Inverse
  - Associative
  - Closure

- Closed (finite intersection of closed sets)
- Bounded $R_{i,j} \in [-1, +1]$
- Does not form a vector space.
- Manifold of dimension $n(n-1)/2$
  - $\text{Dim}(\text{SO}(2)) = 1$
  - $\text{Dim}(\text{SO}(3)) = 3$
Parameterizations of SO(3)

– Even though a rotation matrix is 3x3 with nine numbers, it only has three degrees of freedom. It can be parameterized with three numbers. There are many parameterizations.

• Other common parameterizations
  – Euler Angles
  – Axis Angle
  – Quaternions
  • four parameters; homogeneous
3-D Rotation about the Z axis

\[
\begin{bmatrix}
\tilde{X}' \\
\tilde{Y}' \\
\tilde{Z}'
\end{bmatrix} =
\begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{X} \\
\tilde{Y} \\
\tilde{Z}
\end{bmatrix}
\]

\[\tilde{X}' = R_Z(\gamma)\tilde{X}\]
3-D rotations about X and Y axes

- About X axis:
  \[
  \begin{bmatrix}
  \tilde{X}' \\
  \tilde{Y}' \\
  \tilde{Z}'
  \end{bmatrix} =
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \alpha & -\sin \alpha \\
  0 & \sin \alpha & \cos \alpha
  \end{bmatrix}
  \begin{bmatrix}
  \tilde{X} \\
  \tilde{Y} \\
  \tilde{Z}
  \end{bmatrix}
  \]

  \[\tilde{X}' = R_X(\alpha)\tilde{X}\]

- About Y axis:
  \[
  \begin{bmatrix}
  \tilde{X}' \\
  \tilde{Y}' \\
  \tilde{Z}'
  \end{bmatrix} =
  \begin{bmatrix}
  \cos \beta & 0 & \sin \beta \\
  0 & 1 & 0 \\
  -\sin \beta & 0 & \cos \beta
  \end{bmatrix}
  \begin{bmatrix}
  \tilde{X} \\
  \tilde{Y} \\
  \tilde{Z}
  \end{bmatrix}
  \]

  \[\tilde{X}' = R_Y(\beta)\tilde{X}\]
Euler Angles: Roll-Pitch-Yaw

- Composition of rotations

\[ R = R_Z(\gamma) R_Y(\beta) R_X(\alpha) \]

\[
R = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix}
\]
Coordinate Change: Translation Only

\[
\begin{align*}
\begin{bmatrix}
\tilde{X}' \\
\tilde{Y}' \\
\tilde{Z}'
\end{bmatrix}
&= \begin{bmatrix}
\tilde{X} \\
\tilde{Y} \\
\tilde{Z}
\end{bmatrix}
+ \begin{bmatrix}
t_X \\
t_Y \\
t_Z
\end{bmatrix} \\
\tilde{X}' &= \tilde{X} + t \\
\begin{bmatrix}
\tilde{X}' \\
1
\end{bmatrix}
&= \begin{bmatrix}
I & t \\
0^T & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{X} \\
1
\end{bmatrix} \\
X' &= \begin{bmatrix}
I & t \\
0^T & 1
\end{bmatrix} X
\end{align*}
\]
Coordinate Changes: Rotation and Translation

\[ \tilde{X}' = R \tilde{X} + t \]

\[
\begin{bmatrix}
\tilde{X}' \\
1
\end{bmatrix} = 
\begin{bmatrix}
R & t \\ 
0^\top & 1
\end{bmatrix} 
\begin{bmatrix}
\tilde{X} \\
1
\end{bmatrix}
\]

\[ X' = \begin{bmatrix}
R & t \\ 
0^\top & 1
\end{bmatrix} X \]
What if camera coordinate system differs from world coordinate system?

\[
\begin{align*}
\tilde{X}_{\text{camera}} &= R \tilde{X}_{\text{world}} + t \\
\begin{bmatrix}
\tilde{X}_{\text{camera}} \\
1
\end{bmatrix} &= \begin{bmatrix}
R & t \\
0^T & 1
\end{bmatrix} \begin{bmatrix}
\tilde{X}_{\text{world}} \\
1
\end{bmatrix} \\
X_{\text{camera}} &= \begin{bmatrix}
R & t \\
0^T & 1
\end{bmatrix} X_{\text{world}}
\end{align*}
\]
Intrinsic parameters

\[ K = \begin{bmatrix} \alpha_x & 0 & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \begin{align*}
\alpha_x &= f m_x \\
\alpha_y &= f m_y \\
x_0 &= m_x p_x \\
y_0 &= m_y p_y
\end{align*} \]

focal length in terms of pixel dimensions

principal point in terms of pixel dimensions

\( m_x \) and \( m_y \) are number of pixels per unit distance in \( x \) and \( y \) directions, respectively
Perspective projection camera model

- Extrinsic parameters: Euclidean transformation from the world coordinate frame to the camera coordinate frame
- Intrinsic parameters: Camera calibration matrix embodying focal length and principal point (and pixel aspect ratio and skew)

\[
\begin{bmatrix}
x \\
y \\
w
\end{bmatrix} = \begin{bmatrix}
\alpha_x & 0 & x_0 \\
0 & \alpha_y & y_0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
r_{11} & r_{12} & r_{13} & t_X \\
r_{21} & r_{22} & r_{23} & t_Y \\
r_{31} & r_{32} & r_{33} & t_Z \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z \\
W
\end{bmatrix}
\]

\[x = K[I | 0] \begin{bmatrix}
R \\
0^T \\
1
\end{bmatrix} X\]

\[x = K[R | t]X\]

\[x = PX\]
Deviations from the lens model

Deviations from this ideal are *aberrations*

*Two types*

1. geometrical
   - spherical aberration
   - astigmatism
   - distortion
   - coma

2. chromatic

Aberrations are reduced by combining lenses

*Compound lenses*
Spherical aberration

Rays parallel to the axis do not converge

Outer portions of the lens yield smaller focal lengths
Astigmatism

An optical system with astigmatism is one where rays that propagate in two perpendicular planes have different focus. If an optical system with astigmatism is used to form an image of a cross, the vertical and horizontal lines will be in sharp focus at two different distances.
Distortion

magnification/focal length different for different angles of inclination

Can be corrected! (if parameters are known)
Coma

point off the axis depicted as comet shaped blob
Camera Calibration

- Given $n$ points $P_1, \ldots, P_n$ with known 3-D position and their pixel coordinates $x_1, \ldots, x_n$, estimate intrinsic $K$ and extrinsic camera parameters and lens distortion parameters.
- See textbook for details.
- Camera Calibration Toolbox for Matlab (Bouguet)
  http://www.vision.caltech.edu/bouguetj/calib_doc/
For all cameras?
Other camera models

- Generalized camera – maps points lying on rays and maps them to points on the image plane.

Omnican (hemispherical)  Light Probe (spherical)
Some Alternative “Cameras”
Next Lecture

• Photometric image formation

• Reading:
  – Szeliski
    • Section 2.2