CSE 200 - Winter 2020
Homework 4
Due Wednesday, Dec 1st, 11:59pm

Problem 1: finding a simple path

Let G=(V,E) be an undirected graph with |V|=n, and fix k>0. We saw in class an algorithm based on coloring coding, that checks if G has a simple path of length k. The runtime of the algorithm was $poly(2^k, n)$.

Design a randomized algorithm that finds a simple path of length k whenever one exists. The runtime should still be $poly(2^k, n)$, and the algorithm should succeed with probability at least 50%, say.

Problem 2: finding a bipartite matching

Let G=(U,V,E) be a bipartite graph with |U|=|V|=n. We saw in class an algorithm based on Polynomial Identity Testing (PIT) that decides in poly-time whether G contains a bipartite matching or not.

Design an algorithm that (with high probability) finds a bipartite matching whenever one exists. The runtime should still be poly(n), and the algorithm should succeed with probability at least 50%, say.

Problem 3: Reliably and Probably Useful (RPU) algorithms

We proved in class that ZPP = RP ∩ co-RP. Here, we will define another model of randomization called RPU (Reliably and Probably Useful), which you will prove is also equivalent to ZPP.

An **RPU algorithm** is a randomized algorithm M, that, given an input $x \in \{0,1\}^*$, outputs an answer $M(x) \in \{0,1,?\}$. Here “?” means “I don’t know”. It computes a language $L \subset \{0,1\}^*$ if:

1. It is **reliable:** when the algorithm makes a prediction (outputs 0 or 1) it has to be correct. Namely, if $x \in L$ then $Pr\,[M(x) = 0] = 0$ and if $x \notin L$ then $Pr\,[M(x) = 1] = 0$.
2. It is **useful**: it makes a prediction with some probability on each input. Concretely, for any input x, $Pr\,[M(x) = ?] \leq 1/2$.

Prove that the class of languages that can be computed by an RPU algorithm running in poly-time is the same as ZPP.

Problem 4: PSPACE does not have fixed polynomial size circuits

Recall that

- $PSPACE = \cup_{k\geq 1}\, SPACE(n^k)$ is the class of languages computable in polynomial space
- $P/poly = \cup_{k\geq 1}\, SIZE(n^k)$ is the class of languages computable by polynomial size circuits

We believe that PSPACE is not a subset of P/poly, but this is open. In this question you will prove a weaker statement: PSPACE is not a subset of $SIZE(n^k)$ for any fixed k.

Fix $k \geq 1$. Your goal is to construct a language $L_k \subset \{0,1\}^*$ that satisfies two properties:

(a) L_k can be decided in PSPACE. In fact, it is decided in $SPACE(n^l)$ for some $l = l(k)$.

(b) There exists $n_0 = n_0(k)$, such that for all $n > n_0$ the language $L_k \cap \{0,1\}^n$ cannot be computed by boolean circuits of size n^k.

Steps:

1. Fix an input length n. Let F_n be the class of functions $f: \{0,1\}^n \to \{0,1\}$ which can be computed by a circuit of size n^k. Prove that $|F_n| \leq 2^m$ for $m = O(n^{k+1})$.
2. Let $t \geq 1$ and fix distinct inputs $x_1, \dots, x_t \in \{0,1\}^n$. Prove that there exist values $y_1, \dots y_t \in \{0,1\}$ such that the number of functions $f \in F_n$ that satisfy $f(x_i) = y_i$ is at most 2^{m-t}.
3. Argue that for $t = m + 1$, there are inputs $x_1, \dots, x_t \in \{0,1\}^n$ and values $y_1, \dots y_t \in \{0,1\}$ such that any function $f: \{0,1\}^n \to \{0,1\}$ which satisfies $f(x_i) = y_i$ must be <u>outside</u> F_n.
4. Prove that given an input length n, you can find such inputs and outputs in space poly(m).
5. Complete the proof - describe L_k and prove its properties.