CSE 291-E: Applied Cryptography

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UCSD

Fall 2020 Lecture 17

Legal Notice

The Zoom session for this class will be recorded and made available asynchronously on Canvas to registered students.

Announcements

1. HW 7 is due today!

2. HW 8 is out today! Due in 1.5 weeks!

Last time:

- Number field sieve

This time:

- Lattice-based cryptanalysis

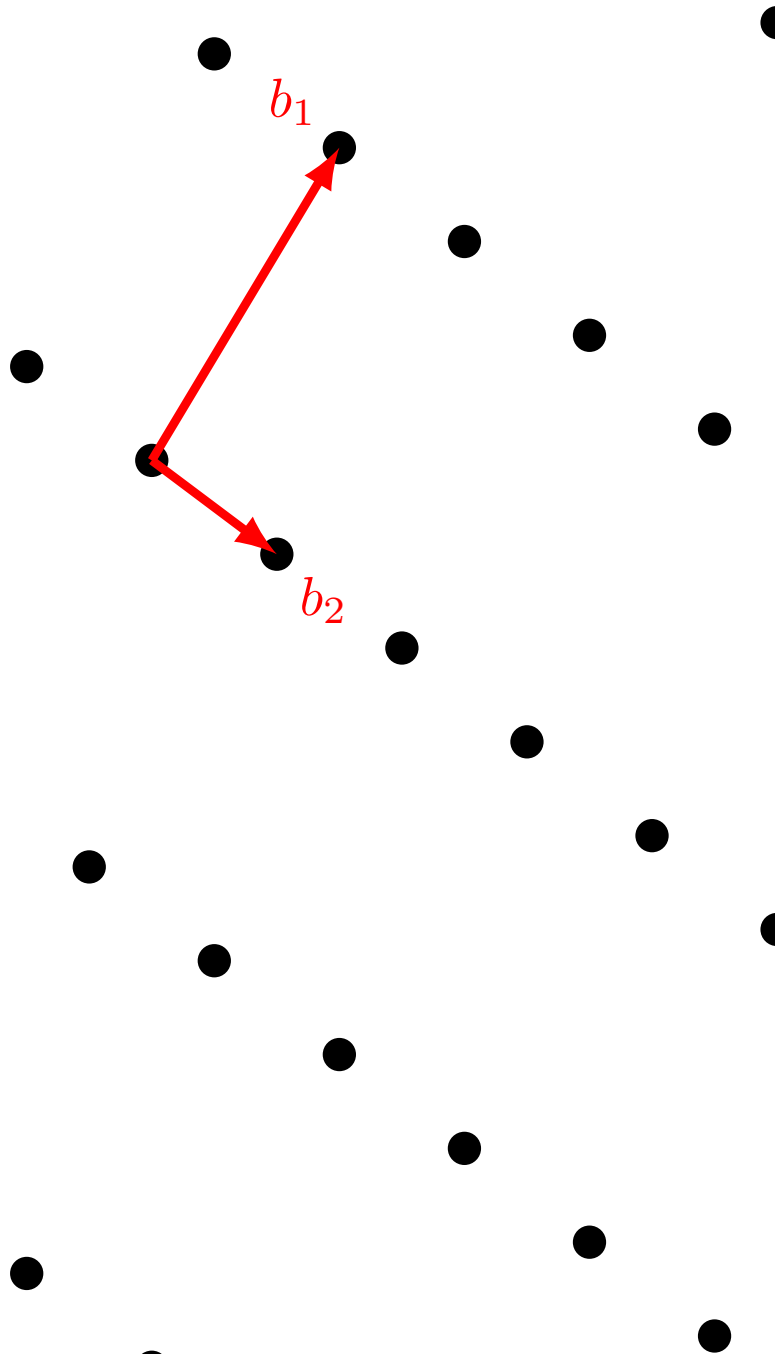
What is a lattice?

Definition

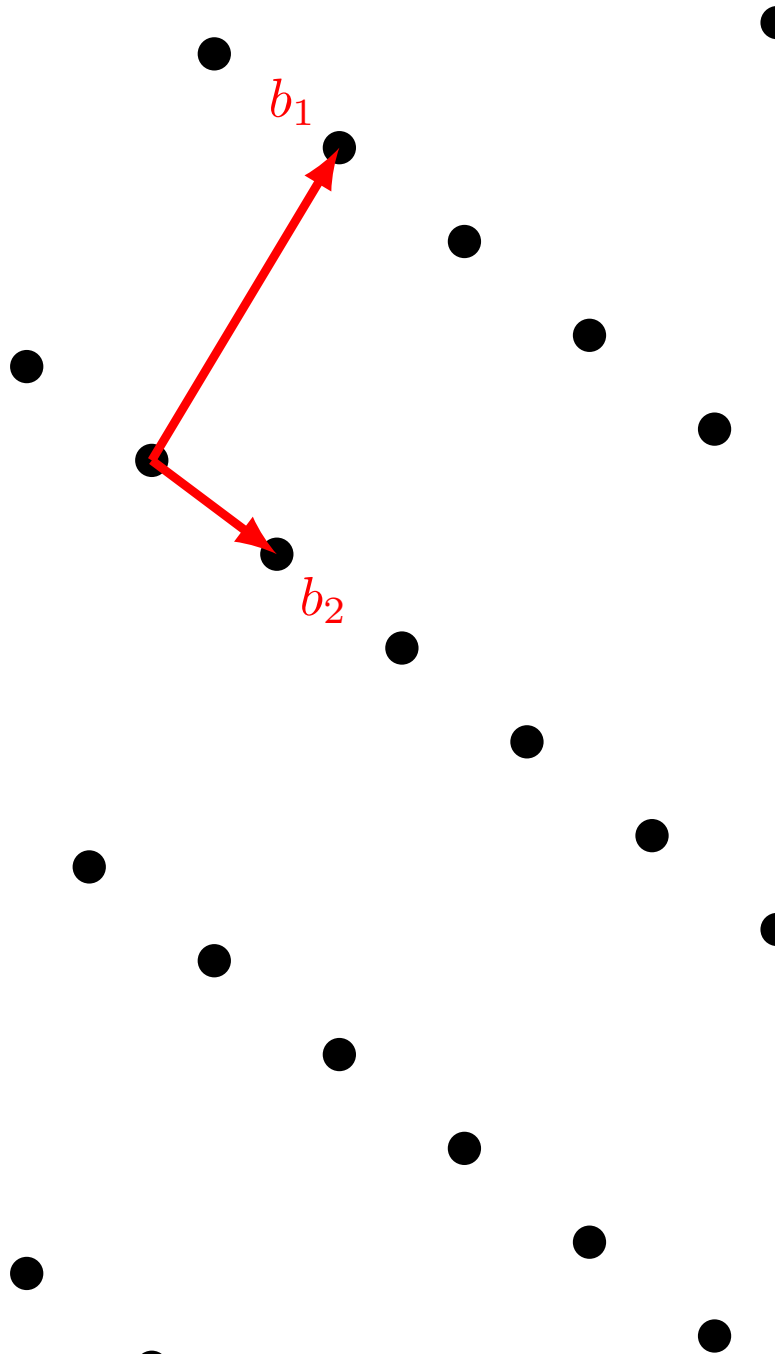
A **lattice** is a subset of $\mathbb{R}^n$ generated by integer linear combinations of some linearly independent basis $\{b_1, \ldots, b_n\}$.

Can represent as Cartesian coordinates: origin $(0, 0, \ldots, 0)$ $b_i = (z_1, \ldots, z_n)$.

- Has algebraic properties (it's a group under addition).
- Has geometric properties (it lives in $\mathbb{R}^n$ so has dot product, distance).



What is a lattice?



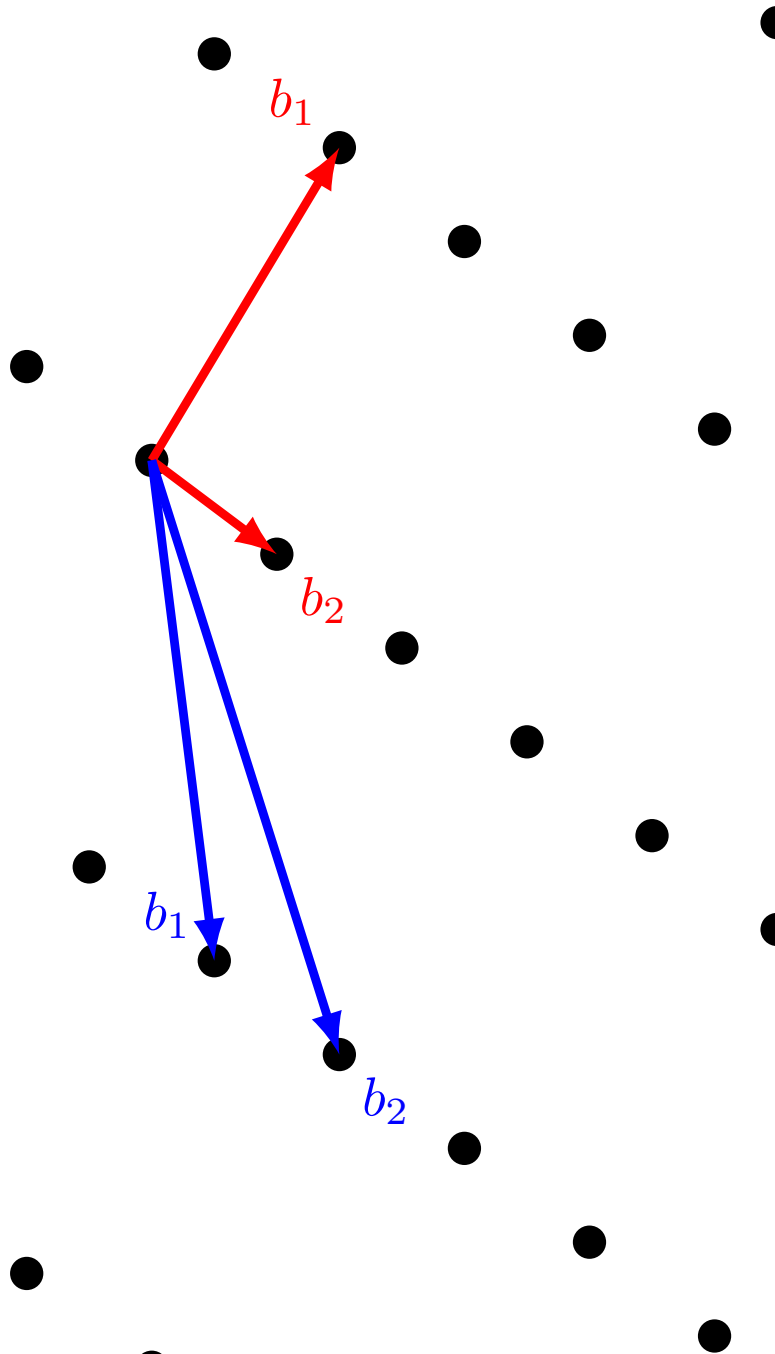
Definition

A **lattice** is a discrete additive subgroup of $\mathbb{R}^n$.

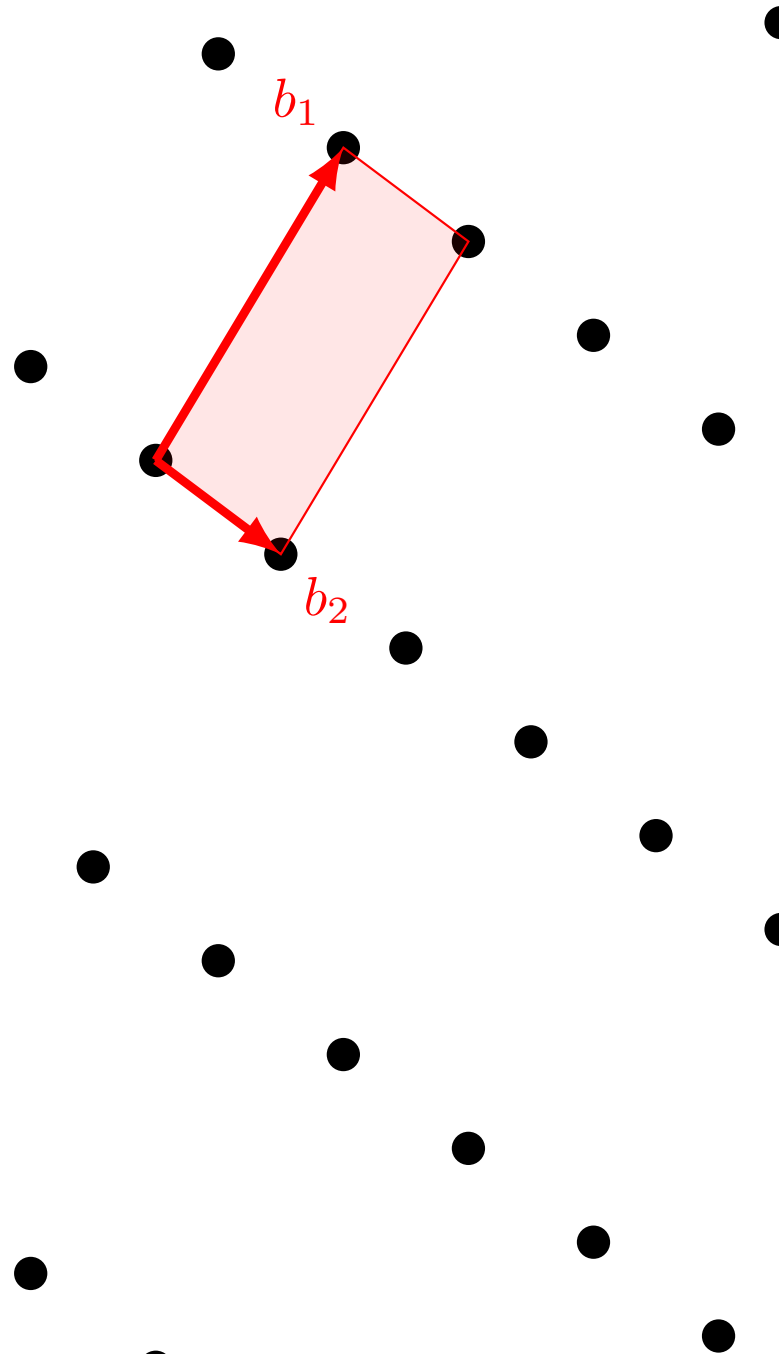
- Discrete: $\exists \delta > 0$ s.t. $|v_i - v_j| > \delta$ $\forall\ v_i, v_j \in L(B)$.
- Additive subgroup: closed under addition.

Properties of lattices: Bases

- In n dimensions a lattice has a basis of size at most n.
- The basis is not unique.
- Let $L(B)$ be the lattice generated by basis B. Deciding if $L(B) = L(B')$ for $B \neq B'$ is efficient. The Hermite Normal Form is unique and efficient to compute. Check if HNF(B) = HNF(B').



Properties of lattices: Determinant



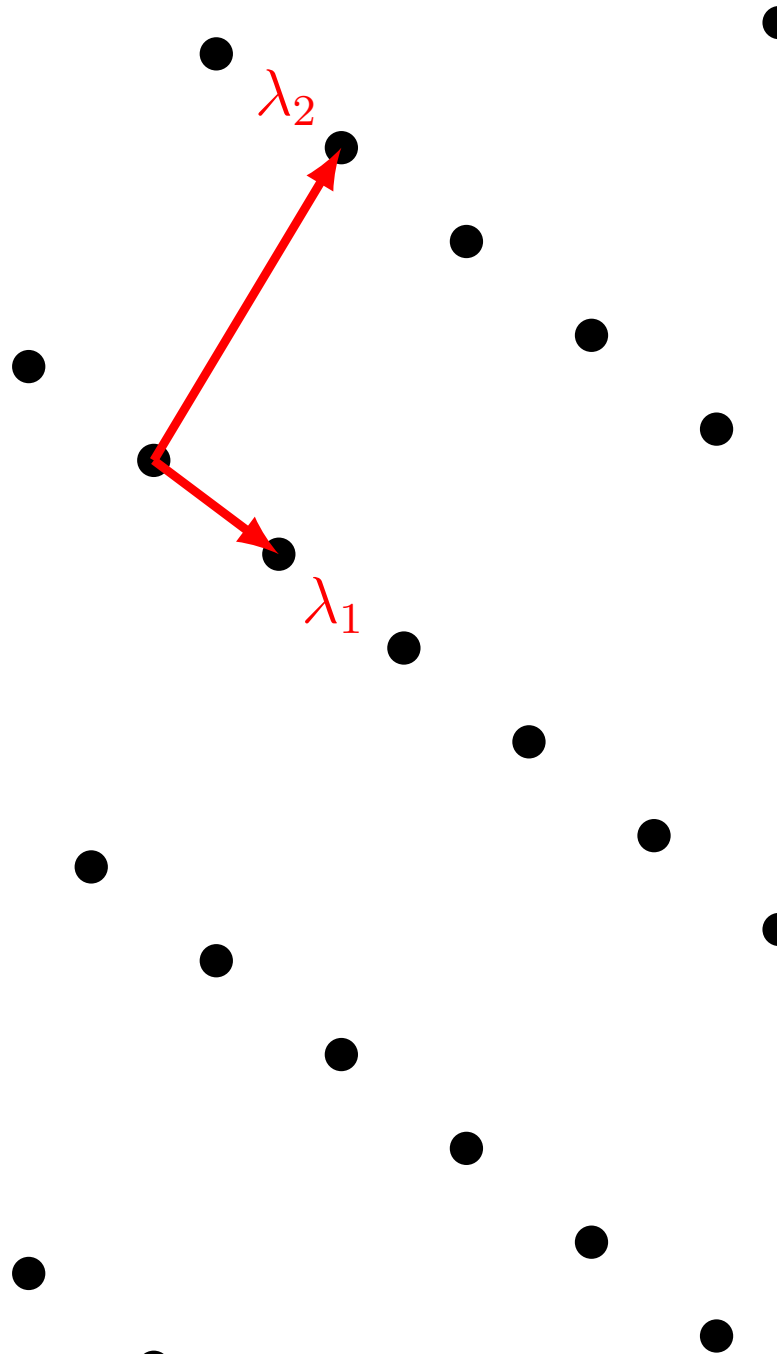
Definition

The **determinant** of a lattice with a basis matrix B is $|\det B|$.

- The determinant is invariant for a given lattice.
- Gives volume of fundamental parallelepiped.

Properties of lattices: Minima

Let $\lambda_1 > 0$ be the length of the shortest vector in the lattice.



Theorem (Minkowski)

$\lambda_1(L) < \sqrt{n} \det L^{1/n}$

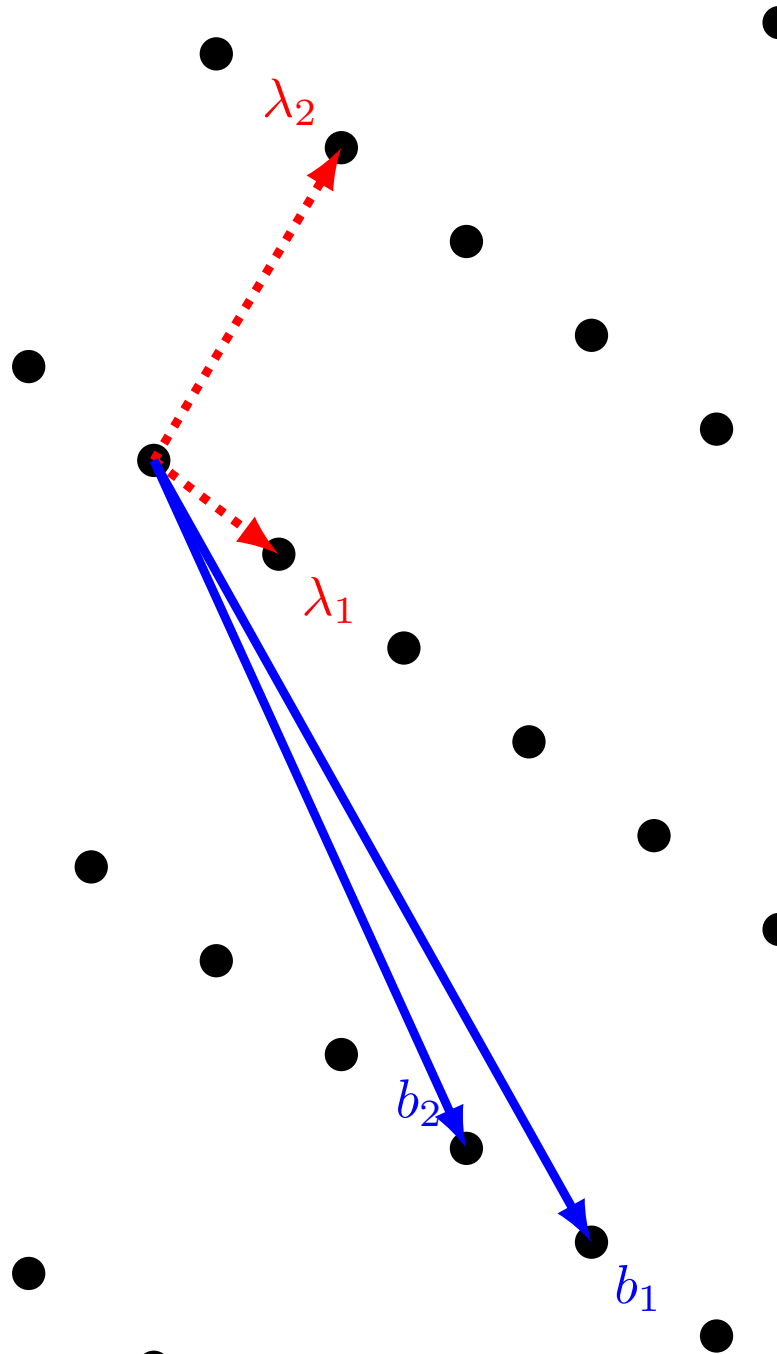
Can define *successive minima* λ_i, the length of the shortest vector linearly independent to the vectors achieving the $i-1$ successive minima.

Computational problems on lattices: SVP

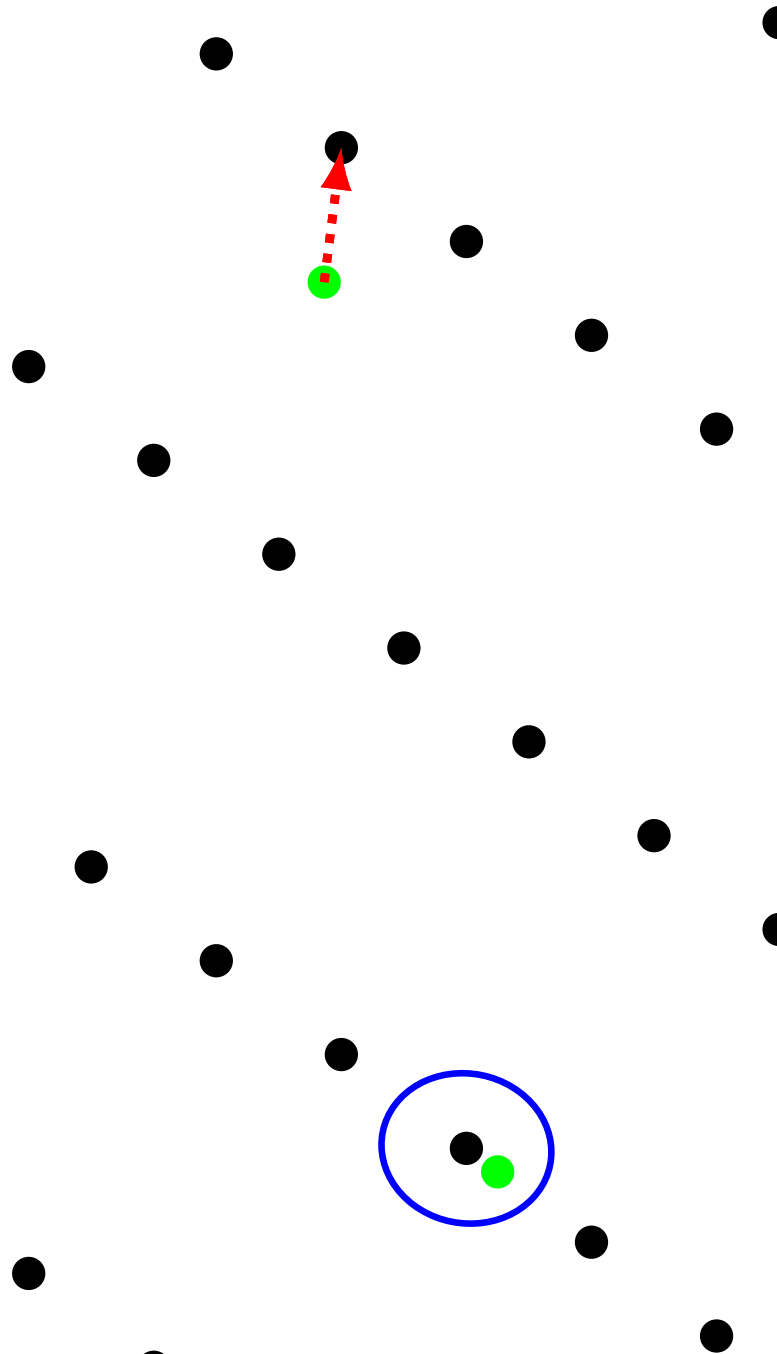
Shortest Vector Problem (SVP)

Given an arbitrary basis for L, find the shortest vector in L.

- SVP is NP-hard.



Computational problems on lattices: CVP



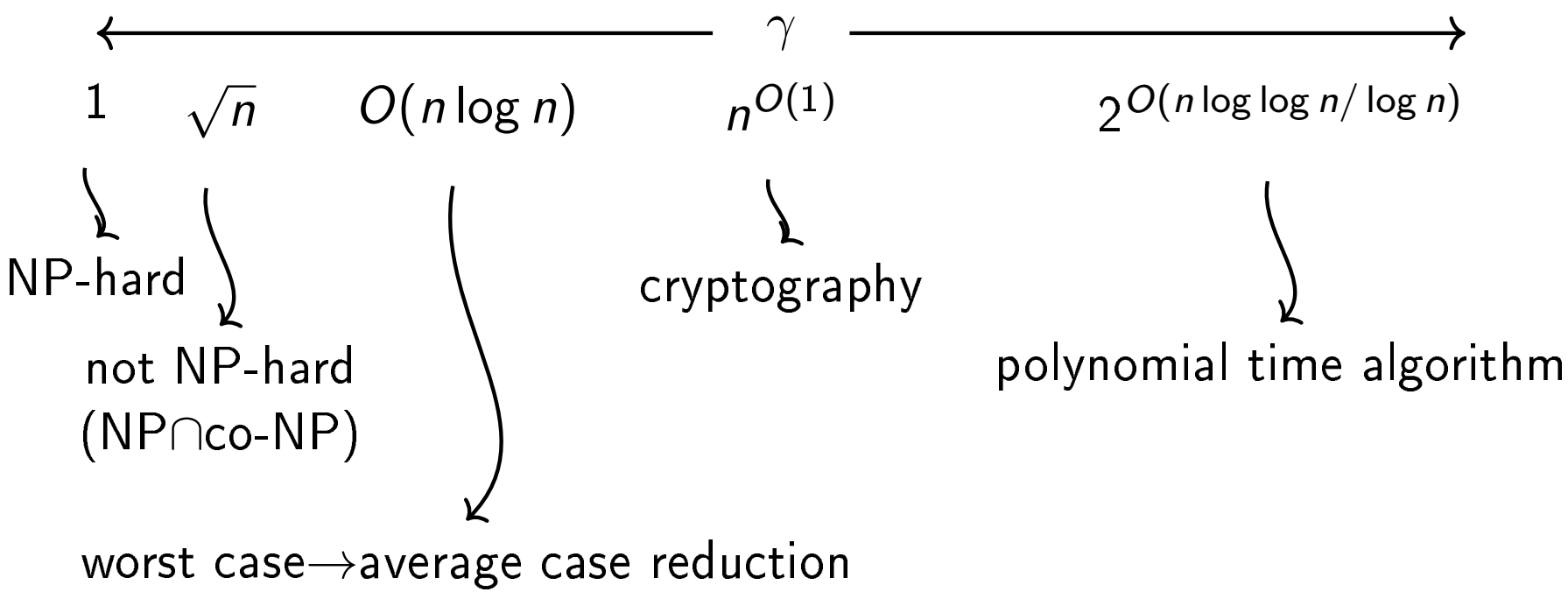
Closest Vector Problem (CVP)

Given an arbitrary basis for L, and a point x find the vector in L closest to x.

- CVP is NP-hard.

Approximation results for SVP

Input: Lattice basis B.
Desired output: Vector of length $\gamma\lambda_1(L(B))$.



Algorithmic results for SVP

Lenstra Lenstra Lovasz (LLL)

Given a basis for a lattice can in polynomial time find a *reduced* basis $\{b_i\}$ s.t.

$$|b_i| \leq 2^{(n-1)/2}\lambda_i$$

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Theorem (LLL (Simplified Version))

We can find a vector of length

$$|v| < 2^{\dim L}(\det L)^{1/\dim L}$$

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- In practice on random lattices, LLL finds $v = 1.02^n(\det L)^{1/\dim L}$. [Nguyen,Stehle]

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BKZ

Given a lattice basis, can in time $2^{O(k)}$ find a reduced basis s.t. $|b_i| \leq k^{O(n/k)}\lambda_i$.

The "two faces" of lattices in cryptography

- **Cryptanalysis:** Can use approximation algorithms for SVP in lattices to cryptanalyze a wide variety of classical cryptography:
  - Attacks on low public exponent RSA
  - Factoring with partial knowledge
  - (EC)DSA with partial information about nonces
  - Knapsack-based cryptosystems

- **Cryptographic constructions:**
  - Lattice problems appear to be hard to solve for quantum computers, so lattice-based cryptosystems among most promising candidates for post-quantum cryptography.
  - Algebraic structure of lattices leads to many interesting cryptographic constructions that may someday be practical, like fully homomorphic encryption, identity-based encryption, etc.

History: Lattices and cryptography

1910	Minkowski's geometry of numbers
1973/1977	Public-key cryptography invented (GCHQ/RSA)
1978	Knapsack cryptography invented (Merkle-Hellmann)
1982	CVP NP-hard (van Emde Boas)
1982	LLL lattice basis reduction algorithm (Lenstra-Lenstra-Lovasz)
1983	LLL algorithm used against knapsack cryptosystems (Lagarias-Odlyzko)
1996	Lattice-based cryptosystems invented (Ajtai-Dwork)
1997	SVP NP-hard (Ajtai)
2005	LWE problem (Regev)
2009	Fully homomorphic encryption using ideal lattices (Gentry)

Historical Interlude: Subset Sum

Subset Sum Problem

Input: Integers $a_1, \ldots, a_n$, target integer T.
Goal: Find a subset $\sum_S a_i = T$.

- NP-hard
- First attempt to base cryptography off of NP-hardness.
- All schemes have a “trapdoor" that lets the decrypter solve the problem. (e.g. super-increasing sequence working modulo some number.)

Solving subset sum with lattices

Input: Integers $a_1, \ldots, a_n$, target integer T.

Generate lattice from rows of matrix

$$\begin{bmatrix} 1 & & & a_1 \\ & 1 & & a_2 \\ & & \ddots & \vdots \\ & & & -T \end{bmatrix}$$

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Generate lattice from rows of matrix

$$\begin{bmatrix} 1 & & & a_1 \\ & 1 & & a_2 \\ & & \ddots & \vdots \\ & & & -T \end{bmatrix}$$

A solution $\sum_i b_i a_i = T$ $b_i \in \{0, 1\}$ determines a vector

$$v = (b_1, b_2, \ldots, 0) \qquad |v|_2 \approx \sqrt{n/2}$$

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$$v = (b_1, b_2, \ldots, 0) \qquad |v|_2 \approx \sqrt{n/2}$$

- $\det L = T$, $\dim L = n + 1$; expect random non-solution vectors to have size $\sqrt{n} \det L^{1/\dim L}$; LLL has approximation factor 1.02^n.
- LLL or BKZ might find short v when $|v| = \sqrt{n/2} < T^{1/(n+1)}$
- Proposed knapsack cryptosystems contained "trapdoors" that made problem easier to solve.

What's wrong with this RSA example?

```
message = Integer('squeamishossifrage',base=35)
N = random_prime(2^512)*random_prime(2^512)
c = message^3 % N
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sage: Integer(c^(1/3)).str(base=35)
'squeamishossifrage'
```

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'squeamishossifrage'
```

The message is too small.

This is why we use padding.

```
N = random_prime(2^150)*random_prime(2^150)
message = Integer('thepasswordfortodayisswordfish',base=35)
c = message^3 % N
```

```
N = random_prime(2^150)*random_prime(2^150)
message = Integer('thepasswordfortodayisswordfish',base=35)
c = message^3 % N

sage: int(c^(1/3))==message
False
```

```
N = random_prime(2^150)*random_prime(2^150)
message = Integer('thepasswordfortodayisswordfish',base=35)
c = message^3 % N
```

This is a stereotyped message. We might be able to guess the format.

```
N = random_prime(2^150)*random_prime(2^150)
message = Integer('thepasswordfortodayisswordfish',base=35)
c = message^3 % N


a = Integer('thepasswordfortodayis000000000',base=35)
```

```
N = random_prime(2^150)*random_prime(2^150)
message = Integer('thepasswordfortodayisswordfish',base=35)
c = message^3 % N


a = Integer('thepasswordfortodayis000000000',base=35)

X = Integer('xxxxxxxxx',base=35)
M = matrix([[X^3, 3*X^2*a, 3*X*a^2, a^3-c],
            [0,N*X^2,0,0],[0,0,N*X,0],[0,0,0,N]])
```

```
N = random_prime(2^150)*random_prime(2^150)
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B = M.LLL()
Q = B[0][0]*x^3/X^3+B[0][1]*x^2/X^2+B[0][2]*x/X+B[0][3]
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B = M.LLL()
Q = B[0][0]*x^3/X^3+B[0][1]*x^2/X^2+B[0][2]*x/X+B[0][3]

sage: Q.roots(ring=ZZ)[0][0].str(base=35)
'swordfish'
```

What's going on here? Coppersmith's method.

Theorem (Coppersmith)

We can efficiently compute up to $1/e$*-fraction of the bits of an RSA-encrypted message with public exponent* e *if we know the rest of the plaintext.*

```
sage: N.nbits()
296
sage: Integer('swordfish',base=35).nbits()
46
```

What's going on here? Coppersmith's method.

Theorem (Coppersmith)

Given a polynomial f of degree d and N, we can efficiently find all roots r_i satisfying

$$f(r_i) \equiv 0 \bmod N$$

when $|r_i| < N^{1/d}$ in time polynomial in $\log N$ and d.

In our case, our input polynomial looks like

$$f(x) = (a + x)^3 - c \equiv 0 \bmod N$$

We are looking for a root $r = \texttt{swordfish}$ satisfying

$$f(r) = (a + \texttt{swordfish})^3 - c \equiv 0 \bmod N$$

Why is this an interesting theorem?

1. A general method to solve polynomials mod N would break RSA: If c is a ciphertext,

$$x^e - c \equiv 0 \bmod N$$

   has a root $x = m$ for m our original message.

2. There is an efficient algorithm to solve equations mod primes.
   - For a composite, factor into primes, solve mod each prime, and use Chinese remainder theorem and Hensel lifting to lift solution mod N.

3. By accepting a bound on solution size, Coppersmith's method lets us solve equations without factoring N.

Coppersmith's Algorithm Outline

Input: polynomial f, modulus N.
Output: a root r modulo N.

In our example, we have $f(x) = (x + a)^3 - c$.

We will construct a new polynomial $Q(x)$ so that

$$Q(r) = 0 \qquad \text{over the integers.}$$

If we construct $Q(x)$ as

$$Q(x) = s(x)f(x) + t(x)N$$

with $s(x), t(x) \in \mathbb{Z}[x]$, then by construction

$$Q(r) \equiv 0 \bmod N$$

(In other words, $Q(x) \in \langle f(x), N \rangle$ over $\mathbb{Z}[x]$.)

Manipulating polynomials

Input: $f(x) = x^3 + f_2x^2 + f_1x + f_0$,N
Output: $Q(x) \in \langle f(x), N \rangle$ over $\mathbb{Z}[x]$.

If we only care about polynomials Q of degree 3, then

$$Q(x) = c_3 f(x) + c_2 N x^2 + c_1 N x + c_0 N$$

with $c_3, c_2, c_1, c_0 \in \mathbb{Z}$.

$$\begin{array}{llllllll}
 & c_3 & (x^3 & + & f_2x^2 & + & f_1x & + & f_0) \\
+ & c_2 & & & Nx^2 & & & & \\
+ & c_1 & & & & & Nx & & \\
+ & c_0 & & & & & & & N \\
\hline
 & & Q_3x^3 & + & Q_2x^2 & + & Q_1x & + & Q_0
\end{array}$$

Manipulating polynomials as coefficient vectors

We can represent elements of $\mathbb{Z}[x]$ as coefficient vectors:

$$g_d x^d + g_{d-1} x^{d-1} + \cdots + g_0 \qquad \leftrightarrow \qquad (g_d, g_{d-1}, \ldots, g_0)$$

If we construct the matrix

$$\begin{bmatrix} 1 & f_2 & f_1 & f_0 \\ & N & & \\ & & N & \\ & & & N \end{bmatrix}$$

Then the coefficient vector representing our polynomial

$$Q(x) = c_3 f(x) + c_2 N x^2 + c_1 N x + c_0 N$$

is an integer combination of the rows of this matrix.

Polynomial coefficient vectors and lattices

The set of vectors generated by integer combinations of the rows of our matrix

$$\begin{bmatrix} 1 & f_2 & f_1 & f_0 \\ & N & & \\ & & N & \\ & & & N \end{bmatrix}$$

is a *lattice*.

Coppersmith's method outline

Input: $f(x) \in \mathbb{Z}[x]$, $N \in \mathbb{Z}$. **Output:** r s.t. $f(r) \equiv 0 \bmod N$.

Intermediate output: $Q(x)$ such that $Q(r) = 0$ over $\mathbb{Z}$.

1. $Q(x) \in \langle f(x), N \rangle$ so $Q(r) \equiv 0 \bmod N$ by construction.

2. If $|r| < R$, then we can bound

$$\begin{aligned} |Q(r)| &= |Q_3 r^3 + Q_2 r^2 + Q_1 r + Q_0| \\ &\leq |Q_3| R^3 + |Q_2| R^2 + |Q_1| R + |Q_0| \end{aligned}$$

3. If $|Q(r)| < N$ and $Q(r) \equiv 0 \bmod N$ then $Q(r) = 0$.

We want a Q in our lattice with short coefficient vector!

Coppersmith's method outline

1. Construct a matrix of coefficient vectors of elements of $\langle f(x), N \rangle$.

2. Run a lattice basis reduction algorithm on this matrix.

3. Construct a polynomial Q from the shortest vector output.

4. Factor Q to find its roots.

Running Coppersmith's method on our example

Input: $f(x) = (x + a)^3 - c$, N
Output: $r < R$ such that $f(r) \equiv 0 \bmod N$.

1. Construct lattice basis

$$\begin{bmatrix} R^3 & 3aR^2 & 3a^2R & a^3 - c \\ & NR^2 & & \\ & & NR & \\ & & & N \end{bmatrix}$$

$\dim L = 4$

$\det L = R^6 N^3$

Factor of R is so that $Q(r) \leq |v|$ for $v \in L$.

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Factor of R is so that $Q(r) \leq |v|$ for $v \in L$.

2. Ignoring approximation factor, we can solve when

$$|Q(r)| \leq |v_1| \leq \det L^{1/\dim L} < N$$
$$(R^6 N^3)^{1/4} < N$$
$$R < N^{1/6}$$

In my example I chose $\lg N = 296$, $\lg r = 46$.

Achieving the Coppersmith bound $r < N^{1/d}$

1. Generate lattice from subset of $\langle f(x), N \rangle^k$.
2. Allow higher degree polynomials.

Theorem (CHHS 2016)

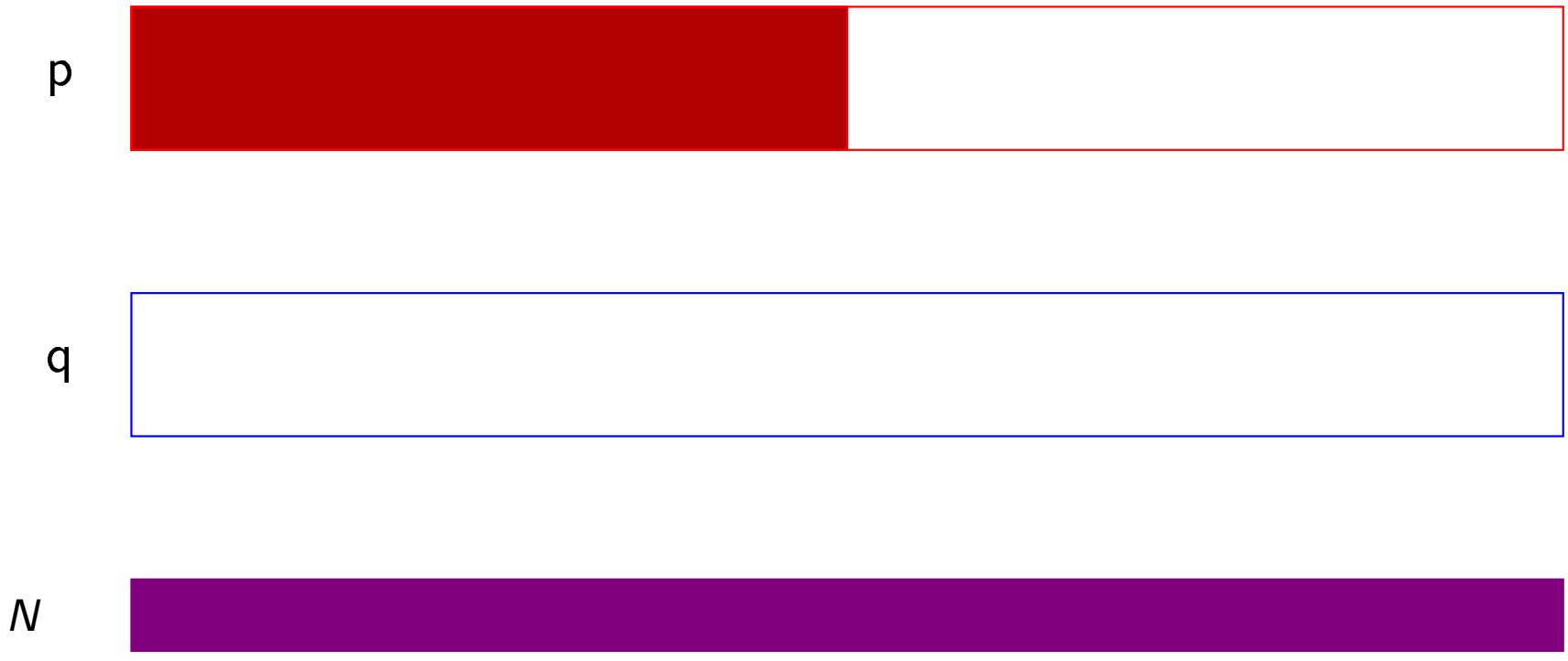
It is not possible to solve for $r > N^{1/d}$ with any method that constructs auxiliary polynomial $Q(x)$.

Countermeasures for real-world RSA

- Must use padding scheme with cryptographically secure randomized padding for RSA.
  - PKCS#1v1.5 widely used in practice, not CCA-secure.
  - OAEP is CCA-secure but not widely used.

- Current recommendation: Use RSA exponent $e \geq 65537$.

Factoring with Partial Information



Polynomial time. (Lattice basis reduction.) [Coppersmith 96]

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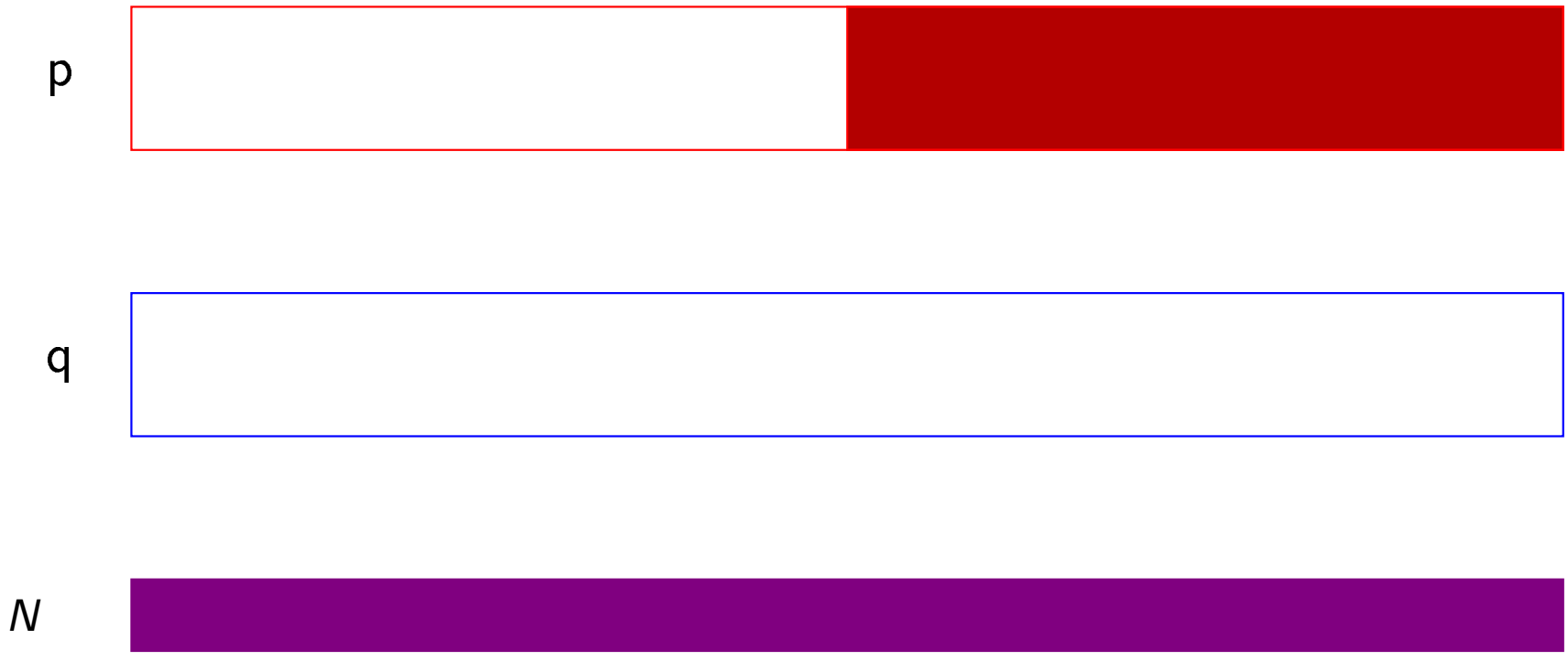

Polynomial time. (Lattice basis reduction.) [Coppersmith 1996]

Theorem (Coppersmith 1996)

Let $N = pq$ with $p, q \approx \sqrt{N}$. Given half the bits (most or least significant) of p, we can factor N in polynomial time.

```
p = random_prime(2^512); q = random_prime(2^512)
N = p*q

a = p - (p % 2^86)
```

```
p = random_prime(2^512); q = random_prime(2^512)
N = p*q

a = p - (p % 2^86)

sage: hex(a)
'a9759e8c9fba8c0ec3e637d1e26e7b88befeb03ac199d1190
76e3294d16ffcaef629e2937a03592895b29b0ac708e79830
4330240bc00000000000000000000'
```

Key recovery from partial information.

```
p = random_prime(2^512); q = random_prime(2^512)
N = p*q

a = p - (p % 2^86)


X = 2^86
M = matrix([[X^2, X*a, 0], [0, X, a], [0, 0, N]])
B = M.LLL()
```

```
p = random_prime(2^512); q = random_prime(2^512)
N = p*q

a = p - (p % 2^86)


X = 2^86
M = matrix([[X^2, X*a, 0], [0, X, a], [0, 0, N]])
B = M.LLL()

Q = B[0][0]*x^2/X^2+B[0][1]*x/X+B[0][2]

sage: a+Q.roots(ring=ZZ)[0][0] == p
True
```

Partial key recovery and finding solutions modulo divisors

Assumption: Attacker doesn't know factorization of N.

Theorem (Howgrave-Graham)

Given degree d polynomial f, integer N, we can find roots r modulo divisors B of N satisfying

$$f(r) \equiv 0 \bmod B$$

for $|B| > N^{\beta}$, when $|r| < N^{\beta^2/d}$ in time polynomial in $\log N$ and d.

For RSA partial key recovery, we have

$$f(x) = a + x \quad d = 1 \quad \beta = 1/2$$

and we want to find a solution vanishing modulo $p \approx N^{1/2}$ for some $p \mid N$.

Efficient to solve for $|r| \leq N^{1/4}$.

Partial key recovery example

Input: $f(x) = a + x, N$
Output: $r < R \quad$ s.t. $\quad f(r) \equiv 0 \bmod p, \quad p|N, \quad p \geq N^{1/2}$

1. We chose the polynomial basis $x(x + a), (x + a), N$.

Partial key recovery example

Input: $f(x) = a + x, N$

Output: $r < R \quad \text{s.t.} \quad f(r) \equiv 0 \bmod p, \quad p|N, \quad p \geq N^{1/2}$

1. We chose the polynomial basis $x(x + a), (x + a), N$.
2. This corresponds to a lattice basis

$$\begin{bmatrix} R^2 & Ra & 0 \\ 0 & R & a \\ & & N \end{bmatrix}$$

$\dim L = 3$

$\det L = R^3 N$

Partial key recovery example

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$$\dim L = 3$$
$$\det L = R^3 N$$

3. LLL will find us a vector of size about $|v| \approx \det L^{1/\dim L}$.

Partial key recovery example

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1. We chose the polynomial basis $x(x + a), (x + a), N$.
2. This corresponds to a lattice basis

$$\begin{bmatrix} R^2 & Ra & 0 \\ 0 & R & a \\ & & N \end{bmatrix} \qquad \begin{aligned} \dim L &= 3 \\ \det L &= R^3 N \end{aligned}$$

3. LLL will find us a vector of size about $|v| \approx \det L^{1/\dim L}$.
4. The algorithm will find the root when we have

$$\begin{aligned} |Q(r)| \leq |v| \approx \det L^{1/\dim L} &< p \\ (R^3 N)^{1/3} &< N^{1/2} \\ R &< N^{1/6} \end{aligned}$$

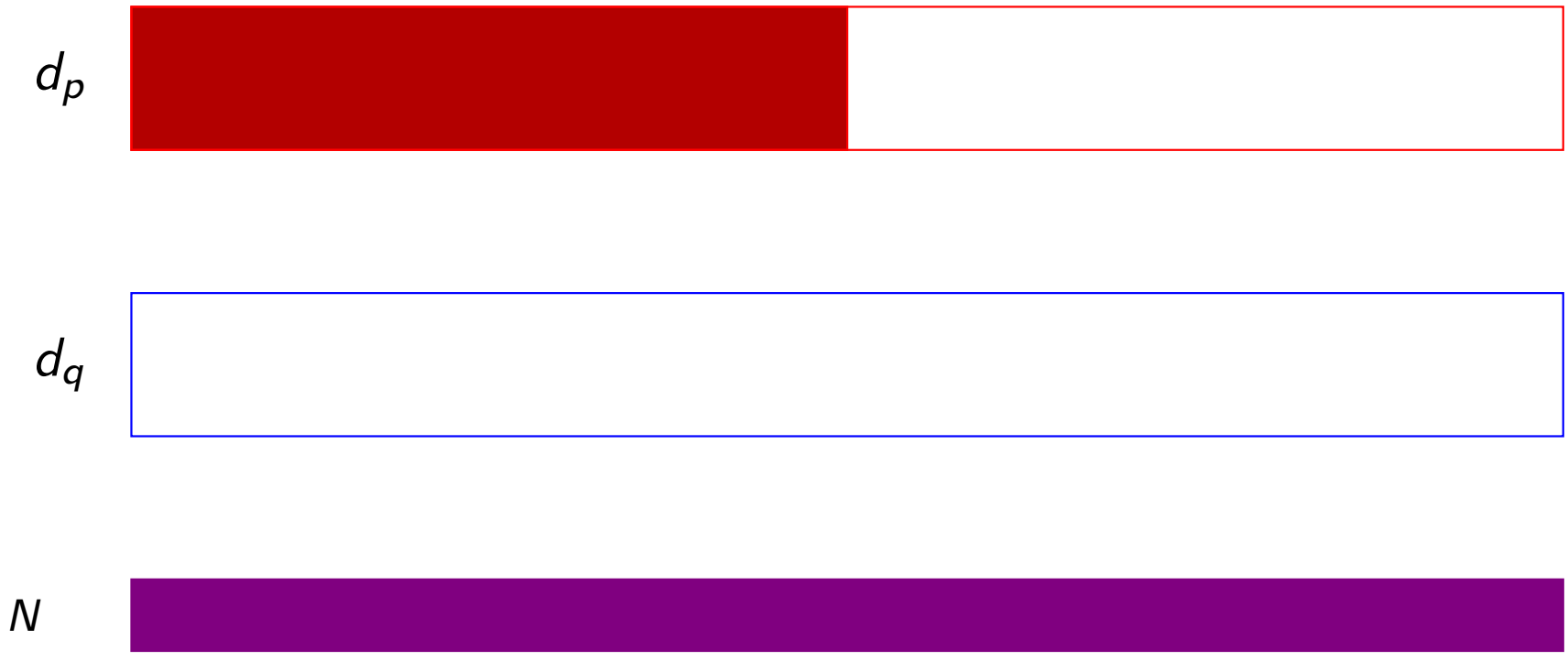
We had $\lg r = 86$ and $\lg p = 512$.

Partial key recovery and related attacks

RSA particularly susceptible to partial key recovery attacks.

- Can factor given 1/2 bits of p. [Coppersmith 96]
- Can factor given 1/4 bits of d. [Boneh Durfee Frankel 98]
- Can factor given 1/2 bits of $d \bmod (p-1)$. [Blömer May 03]

Factoring with Partial Information



Polynomial time. (Lattice basis reduction.) [Blömer May 03]

Key recovery from partial information on CRT-RSA

Let $d_p = d \pmod{p} - 1$.

Assume we know some a such that $d_p = a + r$ and r small.

$$\text{RSA equation:} \qquad ed_p = 1 + k_p(p-1)$$

$$\text{Rearrange:} \qquad (ed_p - 1 + k_p) = k_p p$$

Then we would like to solve for a small solution r to:

$$x + a - e^{-1}(1 + k_p) \equiv 0 \bmod p$$

For e small, we can brute force over k_p, and we know $p|N$.

We can apply Coppersmith/Howgrave-Graham technique as before.

```
p = random_prime(2^512); q = random_prime(2^512)
N = p*q

d = random_prime(2^254)
e = inverse_mod(d,(p-1)*(q-1))
```

d is relatively small. (But not that small.)

```
p = random_prime(2^512); q = random_prime(2^512)
N = p*q

d = random_prime(2^254)
e = inverse_mod(d,(p-1)*(q-1))



X = 2^764; Y = 2^254
M = matrix([[X, e*Y, -1], [0, Y*(N+1), 0], [0, 0, N+1]])

B = M.LLL()
```

```
p = random_prime(2^512); q = random_prime(2^512)
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X = 2^764; Y = 2^254
M = matrix([[X, e*Y, -1], [0, Y*(N+1), 0], [0, 0, N+1]])

B = M.LLL()

sage: abs(B[0][0]/X) == d
True
```

Small RSA private exponent with lattices

Theorem (Wiener)

We can efficiently compute d when $d < N^{1/4}$.

The *RSA equation* is

$$ed \equiv 1 \bmod (p-1)(q-1)$$
$$ed = 1 + k(N - (p+q) + 1)$$

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The *RSA equation* is

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$$ed = 1 + k(N - (p+q) + 1)$$

Let $s = p + q$.

We would like to solve

$$ed = 1 - ks + k(N+1)$$

for d, k, s unknown.

We know $k \leq d$ and $s \approx \sqrt{N}$.

Small RSA private exponent with lattices

We would like to solve

$$ed = 1 - ks + k(N + 1)$$

for d, k, s unknown.

Can write as

$$ks + ed - 1 \equiv 0 \bmod (N + 1)$$

We would like to find small solutions $x = ks, y = d$ for

$$f(x, y) = x + ey - 1 \equiv 0 \bmod (N + 1).$$

Small RSA private exponent with lattices

Would like to solve equation

$$f(x, y) = x + ey - 1 \equiv 0 \bmod (N + 1)$$

for solution $x = ks, y = d$. Bound $|d| < X$, $|ks| < Y$.

Create lattice basis

$$\begin{bmatrix} X & eY & -1 \\ & Y(N+1) & \\ & & (N+1) \end{bmatrix}$$

$\dim L = 3$
$\det L = XY(N+1)^2$

Corresponds to $x + ey - 1$, $y(N + 1)$, $(N + 1)$.

Small RSA private exponent with lattices

We will find a coefficient vector for a polynomial $Q(x, y)$ satisfying $Q(d, ks) = 0$ over $\mathbb{Z}$ when

$$|Q(d, ks)| \leq \dim L^{1/\det L} < N + 1$$
$$(XY(N+1)^2)^{1/3} < N + 1$$
$$XY < N + 1$$

We want to find solutions $d < X$, $ks < Y$, and we know

$$k \leq d \qquad s \approx \sqrt{N}$$

So when $d < X = N^{1/4}$, we can set $Y \approx N^{3/4}$ and $X \approx N^{1/4}$ and guarantee

$$Q(d, ks) = 0.$$

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$$Q(d, ks) = 0.$$

Lattice is actually finding equation

$$dx + (ks - 1)y - d = 0$$

Theorem (Boneh Durfee)

We can efficiently compute d when $d < N^{0.292}$.

Boneh and Durfee use Coppersmith's method to find small solutions $x = k$, $y = (p + q)$ to

$$xy - (N + 1)x - 1 \equiv 0 \bmod e$$

Improvements: Use higher multiplicities and degree, examine sublattice.

ECDSA signature scheme

Public Parameters

- An elliptic curve E
- A base point G of order n on E.

Private Key

- An integer d mod n.

Public Key

- $Q = dG$ in uncompressed (x, y) or compressed $(x, 1 \text{ bit of } y)$ format.

Sign

1. Input message hash h.
2. Choose integer k mod n.
3. Compute point $(r, y_r) = kG$.
4. Output $(r, s = k^{-1}(h + dr) \bmod n)$.

Partial key recovery for (EC)DSA:

An attacker learns some information about the signature nonce k.
Can they efficiently recover the secret key d?

ECDSA key recovery from nonce k: You just did this.

k 

Sign

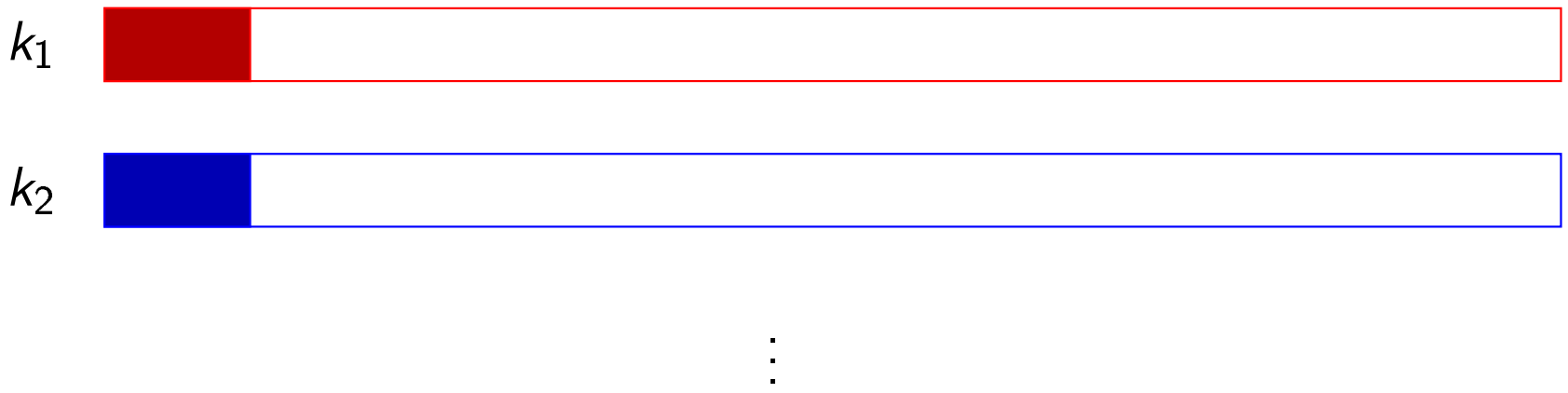
1. Input message hash h.
2. Choose integer k mod n.
3. Compute point $(r, y_r) = kG$.
4. Output $(r, s = k^{-1}(h + dr) \bmod n)$.

Fact

If an attacker learns k for a signature, the long-term secret key d is revealed.

$$d = (sk - h)r^{-1} \bmod n$$

ECDSA key recovery from partial information about nonces



Polynomial time, using lattices. [Howgrave-Graham Smart 2001], [Nguyen Shparlinski 2003]

ECDSA key recovery from partial information about nonces

Secret key d can be computed from MSBs of nonces.

Input signatures $(r_1, s_1), \ldots, (r_m, s_m)$ on messages $h_1, \ldots, h_m$.

Then we have a system of equations in unknowns $k_1, \ldots, k_m, d$:

$$
\begin{aligned}
k_1 - s_1^{-1} r_1 d - s_1^{-1} h_1 &\equiv 0 \bmod n \\
k_2 - s_2^{-1} r_2 d - s_2^{-1} h_2 &\equiv 0 \bmod n \\
&\vdots \\
k_m - s_m^{-1} r_m d - s_m^{-1} h_m &\equiv 0 \bmod n
\end{aligned}
$$

ECDSA key recovery from partial information about nonces

Secret key d can be computed from MSBs of nonces.

Input signatures $(r_1, s_1), \ldots, (r_m, s_m)$ on messages $h_1, \ldots, h_m$.

Assume we have learned MSBs of k_i so that $k_i = a_i + b_i$ with $b_i < B$.

Then we have a system of equations in unknowns $b_1, \ldots, b_m, d$:

$$
\begin{aligned}
b_1 - s_1^{-1} r_1 d + a_1 - s_1^{-1} h_1 &\equiv 0 \bmod n \\
b_2 - s_2^{-1} r_2 d + a_2 - s_2^{-1} h_2 &\equiv 0 \bmod n \\
&\vdots \\
b_m - s_m^{-1} r_m d + a_m - s_m^{-1} h_m &\equiv 0 \bmod n
\end{aligned}
$$

Formulating ECDSA as a hidden number problem

[Howgrave-Graham Smart 2001], [Nguyen Shparlinski 2003]

We have a system of equations in unknowns $b_1, \ldots, b_m, d$:

$$\begin{aligned} b_1 - t_1 d - u_1 &\equiv 0 \bmod n \\ b_2 - t_2 d - u_2 &\equiv 0 \bmod n \\ &\vdots \\ b_m - t_m d - u_m &\equiv 0 \bmod n \end{aligned}$$

We assume the b_i are small.

This is an instance of the *hidden number problem* [Boneh Venkatesan 96].

Solving the hidden number problem with lattices

Input:

$$b_1 - t_1 d - u_1 \equiv 0 \bmod n$$
$$\vdots$$
$$b_m - t_m d - u_m \equiv 0 \bmod n$$

in unknowns $b_1, \ldots, b_m, d$, where $|b_i| < B$.

Construct the lattice

$$M = \begin{bmatrix} n & & & & & \\ & n & & & & \\ & & \ddots & & & \\ & & & n & & \\ t_1 & t_2 & \ldots & t_m & B/n & \\ u_1 & u_2 & \ldots & u_m & & B \end{bmatrix}$$

$v_k = (b_1, b_2, \ldots, b_m, Bd/n, B)$ is a short vector in this lattice.

Solving the hidden number problem with lattices

Construct the lattice

$$M = \begin{bmatrix} n & & & & & \\ & n & & & & \\ & & \ddots & & & \\ & & & n & & \\ t_1 & t_2 & \dots & t_m & B/n & \\ u_1 & u_2 & \dots & u_m & & B \end{bmatrix}$$

Want vector
$v_k = (b_1, b_2, \dots, b_m, Bd/n, B)$

We have:

- $\dim L = m + 2$ $\qquad\qquad$ $\det L = B^2 n^{m-1}$
- Ignoring approximation factors, LLL or BKZ will find a vector

$$|v| \leq (\det L)^{1/\dim L}$$

- We are searching for a vector with length $|v_k| \leq \sqrt{m+2}B$.
- Thus we expect to find v_k when

$$\log B \leq \lfloor \log n(m-1)/m - (\log m)/2 \rfloor$$

Solving the hidden number problem with lattices

We expect to find v_k when

$$\log B \leq \lfloor \log n(m-1)/m - (\log m)/2 \rfloor$$

- 160-bit n: Learn key from 2 bits of information per signature by reducing 100-dimensional lattice [LN 13]
- 256 bit n: Learn key from 3 bits of information per signature by reducing 100-dimensional lattice

Summary

Lattices are a powerful tool especially in side-channel attack scenarios where an attacker can observe partial information about secrets like keys or nonces.