# FHEW: Homomorphic Encryption Bootstrapping in less than a Second[1]

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UC San Diego

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[1]To appear in Eurocrypt 2015
[2]Now at CWI

# Outline

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# The evolution of FHE

Fully Homomorphic Encryption has seen drastic changes since Gentry's first proposal:

- [Rivest,Adleman,Dertouzos'78]: Open problem
- [Gentry'09]: ideal lattices, sparse subset-sum, squashing, etc.
- [Gentry,Halevi'11],[Brakerski,Vaikuntanathan'11]: no squash
- [Brakerski,Vaikuntanathan'11]: Subexponential LWE
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Still, all schemes have a common ingredient:

**Key technique**

Gentry's FHE bootstrapping

# FHE Bootstrapping

All known FHE schemes are based on noisy encryption schemes:

- Decryption is possible only when noise is sufficiently small.
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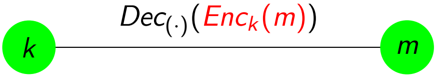
- Idea: homomorphically compute ciphertext decryption function on encrypted key



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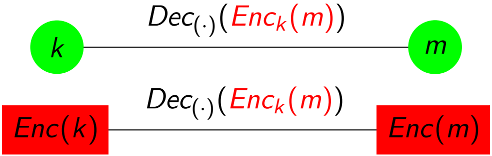

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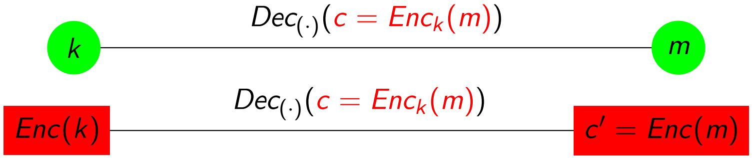

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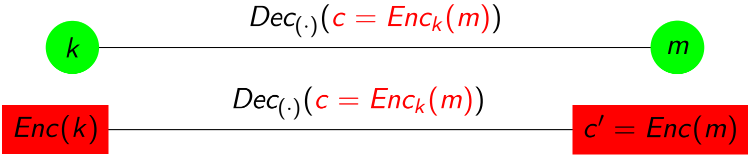


The quality/noise of the output $c'$ depends on

- the quality/noise of $Enc(k)$, which is a fresh ciphertxt, and
- the complexity of $Dec_{(\cdot)}(c)$,
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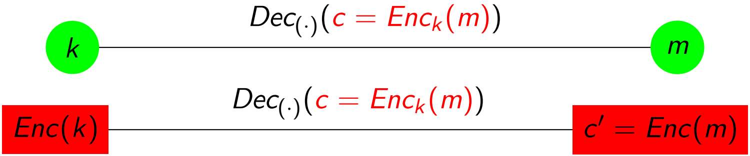


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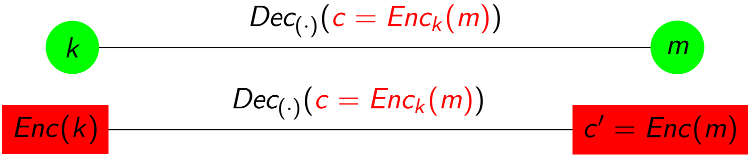


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Lattice Cryptography

- Basic homomorphic properties
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Still, even if $Dec_{(\cdot)}(c)$ is efficient, bootstrapping is very costly because $Dec_{(\cdot)}(c)$ needs to be computed homomorphically on an encrypted $Enc(k)$.

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FHE Bootstrapping/Refreshing is an expensive process:

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- SIMD-FHE: Perform many refresh operations in parallel
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We give a proof of concept solution in 0.6 seconds:
amortized cost comparable to [HElib], but without the delay...
Two new techniques:

- a new, cheap NAND gate
- a simpler refreshing procedure using ring structure

# The new NAND gate

**Base:** Start from LWE encryption with message space: $\mathbb{Z}_t$, $t \geqslant 2$.

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**Idea:** Different message space for input ($t = 4$) and output ($t = 2$).

**Advantages:**

- Cost of computing Homomorphic NAND is negligible (similar to a single private key cryptographic operation.)
- Excellent noise growth: $\epsilon$ grows only by a small constant factor.
- Substantially simplifies the task faced by the Refreshing procedure.

# A simpler refreshing procedure

**Base:** General approach of [Alperin-Sheriff,Peikert'14] + Ring variant of [Gentry,Sahai,Waters'13] Homomorphic encryption.

**Idea:** Implement arithmetic $\bmod q$ **in the exponent**

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**Improvement over [AP14]:**

- Theoretical speed-up of $\tilde{\Omega}(\log^3 q)$
- Smaller final error.

Combined with the problem simpification brought by our cheap NAND computation, this results in bootstrapping cost $\approx 0.6$ second, at estimated $\approx$100-bit security level.

# Outline

Introduction/Summary

The new NAND gate

Simpler Refreshing

Conclusion

# LWE and Symmetric Encryption

**Definition (Learning with Errors)**

- **For** a random secret $\mathrm{s} \in \mathbb{Z}_q^n$
- **Given** many sample $(\mathrm{a}, b = \langle \mathrm{a}\,,\, \mathrm{s} \rangle + e)$ where $e \leftarrow \chi$, small
- **Distinguish** the samples from uniformly random

LWE is as hard as worst-case lattice problems

$$\mathrm{Enc}_{\mathrm{s}}(m \in \mathbb{Z}_2) = (\mathrm{a}, b = \langle \mathrm{a}\,,\, \mathrm{s} \rangle + e + m \cdot q/2)$$
$$\mathrm{Dec}_{\mathrm{s}}(\mathrm{a}, b) = \lfloor 2(b - \langle \mathrm{a}\,,\, \mathrm{s} \rangle)/q \rceil$$

Sets of encryptions of $m$ with error $e < E$ noted $\mathrm{LWE}_{\mathrm{s}}(m, E)$.
**Correct decryption** ensured if $(\mathrm{a}, b) \in \mathrm{LWE}_{\mathrm{s}}(\,\cdot\,, q/4)$

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# Homomorphic Operation on LWE ciphertext

**Addition/XOR operation** as the sum of ciphertexts:

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**(Traditional) (N)AND operation** as the tensor of ciphertexts:

$$\mathsf{LWE}_{\mathsf{s}_1}(m_1, e_1) \times \mathsf{LWE}_{\mathsf{s}_2}(m_2, e_2) \to \mathsf{LWE}_{\mathsf{s}_1 \otimes \mathsf{s}_2}(m_1 \wedge m_2, e_1 \cdot e_2)$$

$\Rightarrow$ FHE boostrapping requires **strong Refreshing** :

$$\mathsf{LWE}_{\mathsf{s}}(m, e) \to \mathsf{LWE}_{\mathsf{s}}(m, e'), \qquad e' \ll e.$$

Techniques: Key Switching, Mod Switching, and Homomorphic Decryption.

# LWE encryption with different message spaces

Idea: use an LWE sample as a mask

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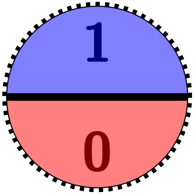
$m \cdot \frac{q}{2} + e$

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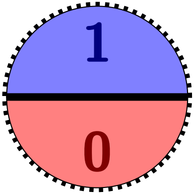


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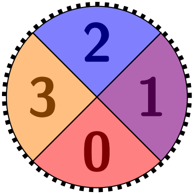


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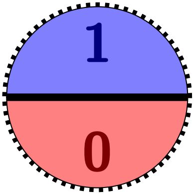


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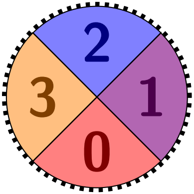


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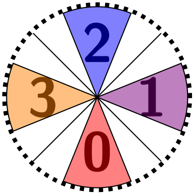
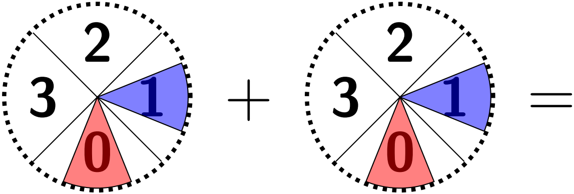



Smaller error
$\mathrm{LWE}_{\mathrm{s}}^{4}(m, q/16)$

# A Cheap NAND gate

**Idea**, use: $m_1 \wedge m_2 \Leftrightarrow m_1 + m_2 = 2 \bmod 4$.
Consider binary messages $\{0, 1\}$ encrypted with $t = 4$:



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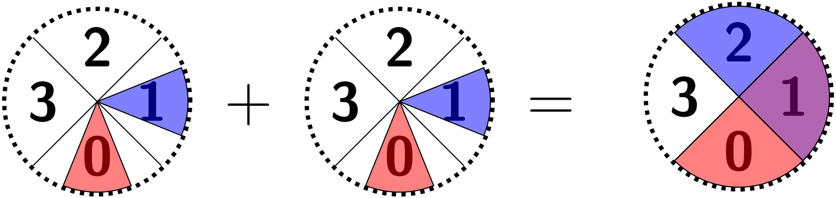

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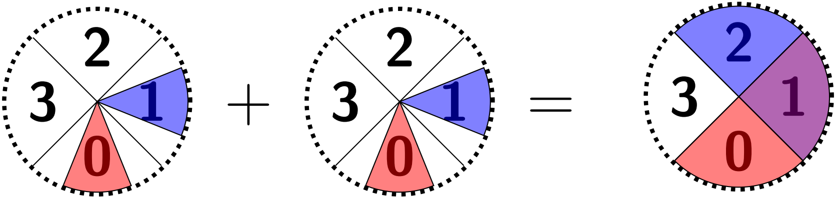


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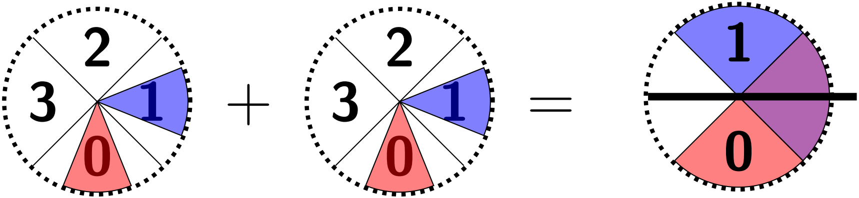


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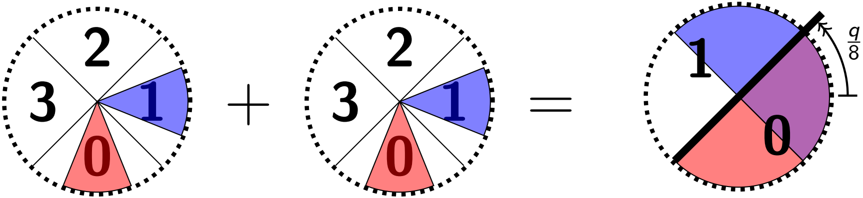


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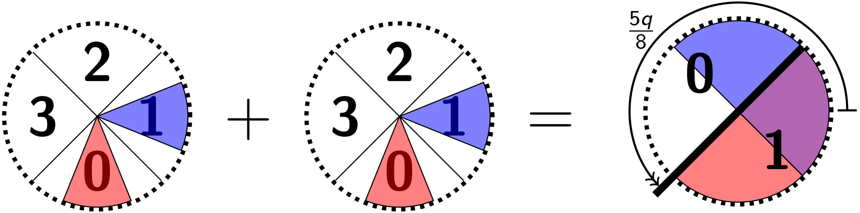


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HomNAND:

$$\mathsf{LWE}_s^4(m_1, \tfrac{q}{16}) \times \mathsf{LWE}_s^4(m_2, \tfrac{q}{16}) \rightarrow \mathsf{LWE}_s^2(m_1 \barwedge m_2, \tfrac{q}{4})$$

$$(\mathrm{a}_1, b_1) \quad , \quad (\mathrm{a}_2, b_2) \mapsto (\mathrm{a}_1 + \mathrm{a}_2, b_1 + b_2 + \tfrac{5q}{8})$$

# Lightweight Refreshing

We have HomNAND:

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To build an FHE we require a relaxed function LightRefresh:

$$\mathsf{LightRefresh} : \mathsf{LWE}^2_{\mathsf{s}}\left(m, q/4\right) \to \mathsf{LWE}^4_{\mathsf{s}}\left(m, q/16\right)$$

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whereas previous works required:

$$\mathrm{Refresh} : \mathrm{LWE}_s^2\left(m, q/4\right) \to \mathrm{LWE}_s^2\left(m, E\right), E \ll q.$$

As usual, we will use Key Switching, Mod Switching, and Homomorphic Decryption.

# Outline

# Decryption using an Accumulator

$$\mathsf{Dec}_{\mathsf{s}}(\mathsf{a}, b) = \mathsf{msb}\left(b - \langle \mathsf{a}\,,\, \mathsf{s} \rangle \bmod q\right) = \mathsf{msb}\left(b - \sum_i a_i \cdot s_i \bmod q\right)$$

**$\mathsf{Dec}_{\mathsf{s}}(\mathsf{a}, b)$:**

$acc \leftarrow b$

for $i = 1$ to $n$:

  $acc \leftarrow acc - a_i \cdot s_i \bmod q$

Return $\mathsf{msb}(acc)$

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**Homomorphic decryption given $E'(\mathsf{s}) = [E(a \cdot s_i) \mid i, a]$:**

$E(acc) \leftarrow b$

for $i = 1$ to $n$:

$\quad E(acc) \leftarrow E(acc) - E(a_i \cdot s_i) \bmod q$

Return LWE(msb($acc$))

$ACC = E(acc)$ holds an encrypted integer $acc \in \mathbb{Z}_q$

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The accumulator ACC should support the following operations:

- Initialization: $ACC \leftarrow b$
- Addition $ACC \leftarrow ACC + c$ of a **fresh ciphertext** $c = E(a \cdot s_i)$
- Extract encrypted MSB: $ACC \rightarrow LWE(msb(acc))$

# Implementing the Accumulator

The framework of [AP14] based on [GSW13] $(E, +, \cdot)$:

- $ACC = [E(a_{q-1}), \ldots, E(a_1), E(a_0)]$ with $a_i = \delta_{i=acc} \in \{0, 1\}$

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We optimize this construction using the cyclotomic ring

$$\mathcal{R} = \mathbb{Z}[X]/(X^N + 1).$$

- Embed $\mathbb{Z}_q$ in the group $(\{X^i\}_i, \cdot)$, of roots of unity, $q = 2N$.
- $ACC$ uses only a single ciphertext $E(X^{acc})$.

# Ring version of [GSW13]

$Q = 2^k$, Gadget matrix: $\mathsf{G} = [\mathsf{I}, 2\mathsf{I}, 4\mathsf{I} \ldots 2^{k-1}\mathsf{I}]^t \in \mathbb{Z}_Q^{n+1\times(n+1)k}$.

$$\mathsf{E}_\mathsf{s}(m) = [\mathsf{A}, \mathsf{A}\mathsf{s} + \mathsf{e}] + m \cdot \mathsf{G}$$

$\mathsf{Dec}_\mathsf{s}$: extract an $\mathsf{LWE}_\mathsf{s}$ ciphertext (last row) and decrypt.
Supports Add. and Mult. for **small messages** $m \in \{-1, 0, 1\}$.

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Cyclotomic ring $\mathcal{R} = \mathbb{Z}[X]/(X^N + 1)$, $\quad 2N = q$ is a power of 2.

Generalized Gadget matrix: $\mathsf{G} = u \cdot [\mathsf{I}, b\mathsf{I}, b^2\mathsf{I} \ldots b^{k-1}\mathsf{I}]^t \in \mathcal{R}_Q^{2 \times 2k}$.

$$\mathsf{E}_\mathsf{s}(m \in \mathbb{Z}_q) = [\mathsf{a}, \mathsf{a} \cdot s + \mathsf{e}] + X^m \cdot \mathsf{G}$$

Supports addition for all message $m \in \mathbb{Z}_q$.

# The group of roots of unity and msb

$$
\begin{array}{r|cccc|cccc}
m & 0 & 1 & \dots & \frac{q}{2}-1 & \frac{q}{2} & \frac{q}{2}+1 & \dots & q-1 \\
X^m & 1 & X & \dots & X^{N-1} & -1 & -X & \dots & -X^{N-1}
\end{array}
$$

Take the vector representation of $X^m$

$$
\begin{array}{r|cccc|cccc}
\mathrm{x}_m & \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} & \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix} & \ddots & \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} & \begin{bmatrix}-1\\0\\\vdots\\0\end{bmatrix} & \begin{bmatrix}0\\-1\\\vdots\\0\end{bmatrix} & \ddots & \begin{bmatrix}0\\0\\\vdots\\-1\end{bmatrix}
\end{array}
$$

Sum all the coordinates

$$
\begin{array}{r|cccc|cccc}
\langle 1\,,\,\mathrm{x}_m\rangle & 1 & 1 & \dots & 1 & -1 & -1 & \dots & -1
\end{array}
$$

$$
\frac{\langle 1\,,\,\mathrm{x}_m\rangle + 1}{2} = \frac{(-1)^{\mathrm{msb}(m)} + 1}{2} = \mathrm{msb}(m).
$$

# Extracting $\mathsf{LWE}^4_s(\mathrm{msb}(m))$

Recall: $\mathsf{G} = u \cdot [\mathsf{I}, b\mathsf{I}, b^2\mathsf{I} \ldots b^{k-1}\mathsf{I}]^t \in \mathcal{R}_Q^{2\times 2k}$ and

$$\mathsf{E}_s(m \in \mathbb{Z}_q) = [\mathsf{a}, \mathsf{a} \cdot s + \mathsf{e}] + X^m \cdot \mathsf{G}$$

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Set $u = q/8$, take the 2nd row, in vector representation:

$$\mathsf{C} = \left[\mathsf{A}', \mathsf{A}' \cdot \mathsf{s} + \mathsf{e} + \frac{Q}{8} \cdot \mathsf{x}_m\right]$$

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Set $u = q/8$, take the $2^{\text{nd}}$ row, in vector representation:

$$\mathsf{C} = \left[\mathsf{A}', \mathsf{A}' \cdot \mathsf{s} + \mathsf{e} + \frac{Q}{8} \cdot \mathsf{x}_m\right]$$

Sum all rows and add $q/8$:

$$\begin{aligned} 1^t \cdot \mathsf{C} = 1^t \cdot & \left[\mathsf{A}', \mathsf{A}' \cdot \mathsf{s} + \mathsf{e} + \frac{Q}{8} \cdot \mathsf{x}_m + \frac{Q}{8}\right] \\ = \quad & \left[\mathsf{a}', \mathsf{a}' \cdot \mathsf{s} + e' + \frac{Q}{4} \cdot \mathsf{msb}(m)\right] \end{aligned}$$

Obtain an LWE encryption of $\mathsf{msb}(m)$ with message space $\mathbb{Z}_4$.

# Improvements

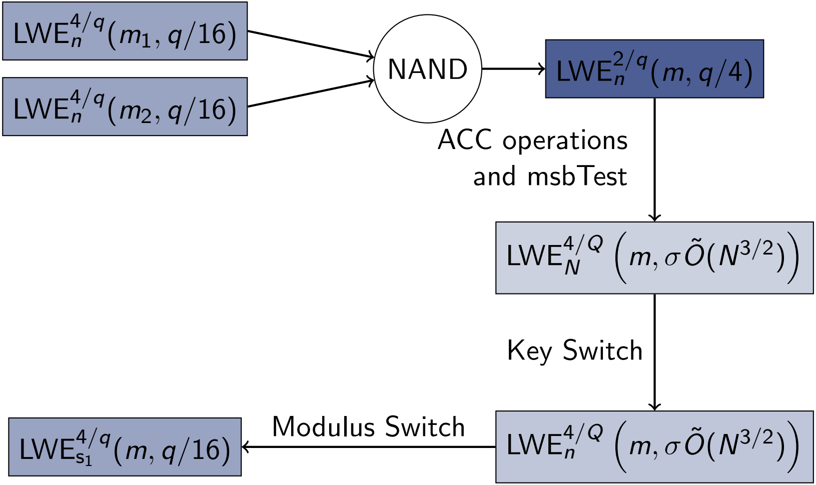
Improvement over the bootstrapping of [AP14]:

- Generic $\tilde{\Omega}(n)$ speed-up from Ring structure
- An extra $\tilde{\Omega}(\log^3 q)$ speed-up by **embedding**
- Error after bootstrapping reduced by $O(\sqrt{n} \log n)$.

In addition to our new NAND gate, implementation becomes reasonable.

# Outline

# The ciphertext cycle



# Parameter Proposal

**Parameters.**

LWE parameters: $n = 410$ $q = 512$.

Ring-GSW parameters: $N = 1024$ $Q = 2^{32}$ .

Gadget Matrix: $Q/8 \cdot [\mathrm{I}, 2^{11} \cdot \mathrm{I}, 2^{22} \cdot \mathrm{I}]$

**Key Size.**

$$\left.\begin{array}{ll}\text{Bootstrapping Key Size} & 846\text{ MB} \\ \text{Key Switching Key Size:} & +135\text{ MB}\end{array}\right\} \leqslant 1\text{GB}$$

**Running time.**

Per NAND gate: $39,360$ FFTs $\approx 0.4$ sec

**Security.**

Security of the LWE scheme $\delta_1 = 1.0060$

Security of the Ring-GSW scheme $\delta_2 = 1.0060$

# Proof of Concept Implementation

- Coded in 4 days·man

  room for implementation level optimization.
- Reasonably concise: $\leqslant 600$ lines of `C++` code

  [HElib]: $\approx 20,000$ lines
- Using FFT[3] over $\mathbb{C}$ at `double` precision

  in dimension 2048 to obtain negacyclic-FFT.

  potentially slower than 32-bits NTT in dimension 1024.

**Result:** Homomorphic NAND & refreshing in 0.61 seconds

on a single standard 64-bit core at 3Ghz.

Comparable to the amortized cost of bootstrapping in [HElib].

[3] FFTW library: The Fastest Fourier Transform in the West

# Beyond binary gates

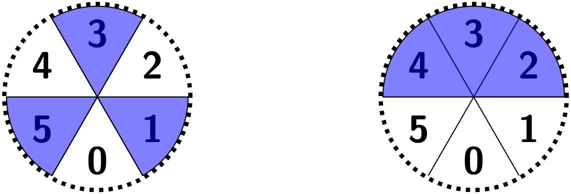
Replace msb by other membership testing function. Obtain :

- moderate fan-in symmetric gates

  majority, threshold gates, **neurons**

- multiple outputs gates

Figure: Full-adder in **one refresh** (3-inputs, 2-outputs)




$m_1 + m_2 + m_3 \in \{1, 3, 5\}$ — $m_1 \oplus m_2 \oplus m_3$

$m_1 + m_2 + m_3 \in \{2, 3, 4\}$ — $\text{carry}(m_1, m_2, m_3)$

# The Next Steps

- Generalize the construction to arbitrary cyclotomic:
  we are restricted to powers of 2
- Exploit previous work using tensor-CRT for coprimes $p, q$:

$$\text{Roots of unity}\big(\mathbb{Z}[X]/\Phi_p(X)\big) \simeq \mathbb{Z}_p$$
$$\text{Roots of unity}\big(\mathbb{Z}[X]/\Phi_p(X) \otimes \mathbb{Z}[X]/\Phi_q(X)\big) \simeq \mathbb{Z}_p \times \mathbb{Z}_q \simeq \mathbb{Z}_{pq}$$

- Obtain a full S-box for the price of 1 or 2 **Refresh** operation.

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# Thank You