CSE 20, Fall 2020 - Homework 7

Due: Monday 11/23 at 11 am PDT

Instructions

Upload a single file to Gradescope for each group. All group members' names and PIDs should be on each page of the submission. You should select appropriate pages for each question when submitting to Gradescope. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Example 5 Section 2.5 (pp. 173-174); Uncountable sets (pp. 173-176);

Key Concepts Uncountable sets, Reals vs Rationals

Problem 1 (20 points)

For each of the following functions, determine if it is one-to-one and/or if it is onto. **Prove your answer**.

1. $f : \mathbb{Z} \to \mathbb{N}$, where $f(n) = |n| + 1$
2. $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $g(n, k) = 3^n \cdot 5^k$
3. $f : \mathbb{N} \to \mathbb{Z}$, where

$$f(n) = \begin{cases} n \text{ div } 4 & \text{if } n \text{ is even} \\ -((n+1) \text{ div } 4) & \text{if } n \text{ is odd} \end{cases}$$

Solution:

1. f is not injective: for example, f(1) = |1| + 1 = |−1| + 1 = f(−1).

   f is not surjective: $f(n) \geq 1$ so there does not exist any $z \in \mathbb{Z}$ such that $f(z) = 0 \in \mathbb{N}$.
2. g is injective: if f(n, k) = f(m, l) then $3^n \cdot 5^k = 3^m \cdot 5^l$, so (n, k) = (m, l) by the fundamental theorem of arithmetic.

   g is not surjective: for example, $2 \neq 3^n \cdot 5^k$ for any $n, k \in \mathbb{N}$, since that would imply 3 divides 2.
3. f is not one to one since f(1) = f(3) = 0. f is onto since for any $z \in F$, if $z \geq 0$, then $f(4z) = z$, else $f(-4z-1) = z$

Problem 2 (20 points)

Find a subset $A \subseteq \mathbb{R}$ for which the function $f : A \to \mathbb{R}$ given by $f(x) = x^2 - 2x + 2$ is one to one. **Prove your answer.**

Solution: We can write $x^2 - 2x + 2 = (x-1)^2 + 1$. Let $A = \{x \in \mathbb{R} : x \geq 1\}$. Then, f is one to one, since for any $x, y \in A$, $f(x) = f(y)$ implies $(x-1)^2 = (y-1)^2$. Since, both $x, y \geq 1$, this gives $x = y$.

Problem 3 (20 points)

The diagonalization argument constructs, for each function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, a set D_f defined as

$$D_f = \{x \in \mathbb{N} \mid x \notin f(x)\}$$

Consider the following two functions with domain $\mathbb{N}$ and codomain $\mathcal{P}(\mathbb{N})$

$$f_1(x) = \{y \in \mathbb{N} \mid y \bmod 3 = x \bmod 3\}$$
$$f_2(x) = \{y \in \mathbb{N} \mid (y > 0) \wedge (x \bmod y \neq 0)\}$$

Select all and only the true statements below.

1. $0 \in D_{f_1}$
2. D_{f_1} is infinite
3. D_{f_1} is uncountable
4. $1 \in D_{f_2}$
5. D_{f_2} is empty
6. D_{f_2} is countably infinite

Solution:

1. False: $0 \in f_1(0)$
2. False: D_{f_1} is empty since $x \in f_1(x) \forall x \in \mathbb{N}$
3. False: D_{f_1} is empty
4. True: $1 \in D_{f_2}$
5. False: D_{f_2} is empty
6. True: D_{f_2} is countably infinite since for all $x \in \mathbb{N}$ and $x > 0$, $x \notin f_2(x)$ and $x \in D_{f_2}$.

Problem 4 (20 points)

Prove the following claim: *"All subsets of a countable set are also countable"*

Solution:
Let set A be a countable set, B be an subset of A. By definition of countable set, there exists an injective function $f : A \to \mathbb{N}$. Define a function $g : B \to \mathbb{N}$ such that $g(x) = f(x), \forall x \in B$. Notice the function g well-defined. Because $\forall x \in B, x \in A$, then $f(x)$ is defined. Also notice g is injective function because $g(x) = f(x) \neq f(y) = g(y), \forall x, y \in B$. Therefore B is also countable because it has an injective function mapped to natural numbers.

Problem 5 (20 points)

Let A be the set of all binary strings of finite length, $A = \cup_{n \geq 1} \{0, 1\}^n$.

Is A countable? **Prove your answer**.

Solution: A is countable. We prove the set of all finite binary strings is countable by showing there is a injective function mapping from set A to the natural numbers. Consider the binary string as the binary expansion of some natural number, than there exists a bijective function f maps the binary number to the natural number, which is also injective. Therefore set A is countable.

Problem 6 - Bonus (10 points)

Prove that any open interval on the real line has the same cardinality.

Solution: Consider a function $g : \mathbb{R} \to (-1, 1)$ defined as $g(x) = \frac{x}{1+|x|}, x \in R$. Notice that this function is bijective, because it is continuous, monotonically increasing and bounded from above and below, so any number on the real line gets mapped to a real number in the the open interval $(-1, 1)$. Therefore, the open interval $(-1, 1)$ has the same cardinality as the real line. Notice that any open interval is a scaled and shifted version of the open interval $(-1, 1)$, therefore there exists a scaled and shifted version of $h(x) = a * g(x) + c, \ a, c \in \mathbb{R}, a \neq 0$, which is also bijective. Hence any open interval on the real line has the same cardinality.