CSE 20, Fall 2020 - Homework 7

Due: Monday 11/30 at 11 am PDT

Instructions

Upload a single file to Gradescope for each group. All group members' names and PIDs should be on each page of the submission. You should select appropriate pages for each question when submitting to Gradescope. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Example 5 Section 2.5 (pp. 173-174); Uncountable sets (pp. 173-176);

Key Concepts Uncountable sets, Reals vs Rationals

Problem 1 (20 points)

For each of the following functions, determine if it is one-to-one and/or if it is onto. **Prove your answer**.

1. $f : \mathbb{Z} \to \mathbb{N}$, where $f(n) = |n| + 1$
2. $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $g(n, k) = 3^n \cdot 5^k$
3. $f : \mathbb{N} \to \mathbb{Z}$, where

$$f(n) = \begin{cases} n \text{ div } 4 & \text{if } n \text{ is even} \\ -((n+1) \text{ div } 4) & \text{if } n \text{ is odd} \end{cases}$$

Problem 2 (20 points)

Find a subset $A \subseteq \mathbb{R}$ for which the function $f : A \to \mathbb{R}$ given by $f(x) = x^2 - 2x + 2$ is one to one. **Prove your answer.**

Problem 3 (20 points)

The diagonalization argument constructs, for each function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, a set D_f defined as

$$D_f = \{x \in \mathbb{N} \mid x \notin f(x)\}$$

Consider the following two functions with domain $\mathbb{N}$ and codomain $\mathcal{P}(\mathbb{N})$

$$f_1(x) = \{y \in \mathbb{N} \mid y \bmod 3 = x \bmod 3\}$$
$$f_2(x) = \{y \in \mathbb{N} \mid (y > 0) \wedge (x \bmod y \neq 0)\}$$

Select all and only the true statements below.

1. $0 \in D_{f_1}$
2. D_{f_1} is infinite
3. D_{f_1} is uncountable
4. $1 \in D_{f_2}$
5. D_{f_2} is empty
6. D_{f_2} is countably infinite

Problem 4 (20 points)

Prove the following claim: *"All subsets of a countable set are also countable"*

Problem 5 (20 points)

Let A be the set of all binary strings of finite length, $A = \cup_{n \geq 1} \{0, 1\}^n$.
Is A countable? **Prove your answer**.

Problem 6 - Bonus (10 points)

Prove that any open interval on the real line has the same cardinality.