CSE 20, Fall 2020 - Homework 6 Solutions

Due: Monday 11/23 at 11 am PDT

Instructions

Upload a single file to Gradescope for each group. All group members' names and PIDs should be on each page of the submission. You should select appropriate pages for each question when submitting to Gradescope. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Section 4.3 Example 2 Section (p. 258), Section 1.7 Example 9 (p. 86); Definitions 1,2,5,7,8 Section 2.3; Definition 3, Example 1 Section 2.5 (p. 171)

Key Concepts Primes and Rationals; Cardinality; Countably infinite sets

Problem 1 (20 points)

Given a sequence of n integers:

$$x_1, x_2, ..., x_n$$

Define the average function as

$$avg(x_1, x_2, ..., x_n) = (x_1 + x_2 + ... + x_n)/n$$

Prove that at least one integer in the sequence is less than or equals to the average. In other words, prove $\exists x_i \ (x_i \leq avg(x_1, x_2, ..., x_n))$ where $1 \leq i \leq n, \ i \in \mathbb{Z}$. Please identify your proof strategy.

Solution: We prove by contradiction.
Assuming towards contradiction, we negate the claim and get $\forall x_i \ (x_i > avg(x_1, x_2, ..., x_n))$.
Summing up all x_i, we have $x_1 + x_2 + ... + x_n > \ avg(x_1, x_2, ..., x_n) * n$. Substituting the definition of average function, we have that $avg(x_1, x_2, ..., x_n) * n = x_1 + x_2 + ... + x_n$, which is not strictly greater than itself. Therefore we have a contradiction, proving the assumption is wrong. Then our original claim is proved true by contradiction.

Problem 2 (20 points)

Prove or disprove the following claim, identify your proof strategy:

"If $1/x$ is irrational, then x is irrational"

Does it also hold for the other direction? i.e. If x is irrational, then $1/x$ is irrational. No proof is needed, just true or false.

Solution: we prove by contrapositive.
Notice that the contrapositive of original claim is "If x is not irrational, then 1/x is not irrational", which is equivalent to "If x is rational, then 1/x is rational". By the definition of rational numbers, we can write x=a/b where a and b are integers. then 1/x=b/a, which is a rational number.
Therefore we proved its contrapositive is true. The original claim was proved by contraposition.
Notice that x can be 0 when a is 0. However, divide by 0 is undefined, so the predicate "1/x is irrational" doesn't make sense. Because it is a conditional claim, the conditional holds true trivially.
And it holds for the other direction as well. i.e. claim, "If x is irrational, then $1/x$ is irrational." is True

Problem 3 (20 points)

(1) Consider set A, B and C, where we have $f: A \rightarrow B$ is one-to-one, and $g: B \rightarrow C$ is bijective.
Fill in the blank with $\leq, =$, or $\geq$

(a) $|A|$ _$\leq$_ $|B|$

(b) $|B|$ _$=$_ $|C|$

(c) $|A|$ _$\leq$_ $|C|$

(2) Define a function operator $\circ$ as $g \circ f(x) = g(f(x))$. Then prove or disprove the following claim:

"$g \circ f: A \rightarrow C$ *is one-to-one*"

Solution: We prove the claim by universal generalization. We show that $g \circ f$ is one-to-one by showing that $\forall a_1, a_2 \in A, a_1 \neq a_2 \rightarrow g \circ f(a_1) \neq g \circ f(a_2)$ Notice there is the function f is one-to-one, by definition of one-to-one function $\forall a_1, a_2 \in A, a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$, and function g is bijective. Therefore we have $\forall b_1, b_2 \in B, b_1 \neq b_2 \rightarrow g(b_1) \neq g(b_2)$. Hence applying function g after f, we have

$\forall a_1, a_2 \in A, (b_1 = f(a_1) \wedge b_2 = f(a_2)) \rightarrow (g(f(a_1)) \neq g(f(a_2)) = g \circ f(a_1) \neq g \circ f(a_2)$

By above we shown the function $g \circ f: A \rightarrow C$ is one-to-one

Problem 4 (20 points)

Consider the set $U = P(R)$, where $P()$ denotes the power set of a set and R denotes the set for real numbers. Prove or disprove the following claims:

(1) $\forall X \in U \ \forall Y \in U \ ((|X| = |Y|) \rightarrow (X = Y))$

(2) $\exists A \in U \ \exists B \in U \ (Z \subseteq A \wedge Z \subseteq B \wedge \neg (|A| = |B|))$, where Z denotes the set of integers.

Solution:

1. This statement is false. Suppose $X = \{1, 2\}$, $Y = \{3, 4\}$. Then we have $|X| = |Y|$. Obviously $X \neq Y$ in this case.
2. This statement is true. We can find such A and B: let $A = Z$, $B = R$, then $(Z \subseteq A \wedge Z \subseteq B)$. Since A is countable and B is uncountable, we have $|A| \neq |B|$

Problem 5 (20 points)

Show that if A and B are sets such that $|A| = |B|$, then $|P(A)| = |P(B)|$.

Solution:

Since $|A| = |B|$, we know that there exists a bijection from A to B. Then there exists a $f(x)$ such that
$\forall a \in A, f(a) \in B$ and $(a_1 \neq a_2) \rightarrow (f(a_1) \neq f(a_2))$
$\forall \{a_1, a_2, ..., a_n\} \in P(A),\ \exists b_1, b_2, ..., b_n \in B\ s.t.\ \{b_1, b_2, ..., b_n\} = \{f(a_1), f(a_2), ..., f(a_n)\} \in P(B)$
Therefore $|P(A)| \leq |P(B)|$.
Similarly, we have $|P(B)| \leq |P(A)|$. Thus, we must have $|P(A)| = |P(B)|$.

Problem 6 - Bonus (10 points)

3 lemmas are presented in the slides for Nov.16's lecture. This time, your job for the bonus problem will be proving these lemmas:

Lemma 1: For every two integers p and q, not both zero, $gcd\left(\frac{p}{gcd(p,q)}, \frac{q}{gcd(p,q)}\right) = 1$.

Lemma 2: For every two integers a and b, not both zero, with $gcd(a, b) = 1$, it is not the case that both a is even and b is even.

Lemma 3: For every integer x, x is even if and only if x^2 is even.

Solution:

1. Suppose $gcd(\frac{p}{gcd(p,q)}, \frac{q}{gcd(p,q)}) = r$. Then $gcd(p,q) \cdot r$ is also a common divisor of p and q. Since p and q are not both zero, we have $gcd(p,q) > 0$. Since $gcd(p,q)$ is the greatest common divisor, we must have $gcd(p,q) \cdot r = gcd(p,q)$, thus $r = 1$.
2. If both a and b are even, then 2 should be a common divisor for a and b, so $gcd(a,b) > 2$, which leads to a contradiction.
3. $\Rightarrow$: if x is even, then x^2 is the product of an even number times the other even number, which is also even.
   $\Leftarrow$: if x is odd, then x^2 is the product of an odd number times the other odd number, which is also odd. Thus, x is odd implies that x^2 can't be even. Thus, x^2 is even implies x is also even.