CSE 20, Fall 2020 - Homework 6

Due: Monday 11/23 at 11 am PDT

Instructions

Upload a single file to Gradescope for each group. All group members' names and PIDs should be on each page of the submission. You should select appropriate pages for each question when submitting to Gradescope. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Section 4.3 Example 2 Section (p. 258), Section 1.7 Example 9 (p. 86); Definitions 1,2,5,7,8 Section 2.3; Definition 3, Example 1 Section 2.5 (p. 171)

Key Concepts Primes and Rationals; Cardinality; Countably infinite sets

Problem 1 (20 points)

Given a sequence of n integers:

$$x_1, x_2, ..., x_n$$

Define the average function as

$$avg(x_1, x_2, ..., x_n) = (x_1 + x_2 + ... + x_n)/n$$

Prove that at least one integer in the sequence is less than or equals to the average. In other words, prove $\exists x_i \ (x_i \le avg(x_1, x_2, ..., x_n))$ where $1 \le i \le n,\ i \in \mathbb{Z}$. Please identify your proof strategy.

Problem 2 (20 points)

Prove or disprove the following claim, identify your proof strategy:

"If $1/x$ is irrational, then x is irrational"

Does it also hold for the other direction? i.e. If x is irrational, then $1/x$ is irrational. No proof is needed, just true or false.

Problem 3 (20 points)

(1) Consider set A, B and C, where we have $f : A \to B$ is one-to-one, and $g : B \to C$ is bijective.
Fill in the blank with $\le, =$, or $\ge$
   (a) $|A|$ ___ $|B|$
   (b) $|B|$ ___ $|C|$
   (c) $|A|$ ___ $|C|$

(2) Define a function operator $\circ$ as $g \circ f(x) = g(f(x))$. Then prove or disprove the following claim:

"$g \circ f : A \to C$ is one-to-one"

Problem 4 (20 points)

Consider the set $U = P(R)$, where $P()$ denotes the power set of a set and R denotes the set for real numbers. Prove or disprove the following claims:

(1) $\forall X \in U \; \forall Y \in U \; ((|X| = |Y|) \rightarrow (X = Y))$

(2) $\exists A \in U \; \exists B \in U \; (Z \subseteq A \land Z \subseteq B \land \neg (|A| = |B|))$, where Z denotes the set of integers.

Problem 5 (20 points)

Show that if A and B are sets such that $|A| = |B|$, then $|P(A)| = |P(B)|$.

Problem 6 - Bonus (10 points)

3 lemmas are presented in the slides for Nov.16's lecture. This time, your job for the bonus problem will be proving these lemmas:

Lemma 1: For every two integers p and q, not both zero, $gcd\left(\frac{p}{gcd(p,q)}, \frac{q}{gcd(p,q)}\right) = 1$.

Lemma 2: For every two integers a and b, not both zero, with $gcd(a,b) = 1$, it is not the case that both a is even and b is even.

Lemma 3: For every integer x, x is even if and only if x^2 is even.