CSE 20, Fall 2020 - Homework 5

Due: Monday 11/16 at 11 am PDT

Instructions

Upload a single file to Gradescope for each group. All group members' names and PIDs should be on each page of the submission. You should select appropriate pages for each question when submitting to Gradescope. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Section 1.5 Table 1 (p. 60); Section 2.1, Definitions 1-3 (pp. 116-119), Definitions 6-8 (pp. 121-123); Section 2.2 Definitions 1-5 (pp. 127-129) and Table 1 (p. 130), Section 5.3 Definition of Structural Induction

Key Concepts Proof Strategies, Set Definitions and Induction

Problem 1 (20 points)

Use mathematical induction to show that
$1^3 + 3^3 + 5^3 + ... + (2n+1)^3 = (n+1)^2(2n^2 + 4n + 1)$
whenever n is a positive integer.

Solution:
When $n = 1$, we have $1^3 + 3^3 = (1+1)^2(2 \cdot 1^2 + 4 \cdot 1 + 1)$
Suppose the equation holds when $n = k$. Then we have
$1^3 + 3^3 + ... + (2 \cdot (k+1) + 1)^3 = (k+1)^2(2k^2 + 4k + 1) + (2 \cdot (k+1) + 1)^3$
$= (2k^4 + 8k^3 + 11k^2 + 6k + 1) + (8k^3 + 36k^2 + 54k + 27) = 2k^4 + 16k^3 + 47k^2 + 60k + 28 = (k+2)^2(2k^2 + 8k + 7)$
$= ((k+1)+1)^2(2 \cdot (k+1)^2 + 4 \cdot (k+1) + 1)$
Which implies the equation holds when $n = k + 1$. Therefore, the equation holds whenever n is a positive integer.

Problem 2 (20 points)

For each part of this problem, clearly give a witness or counterexample that would be appropriate. You do not need to justify your answer. However, if you include clear explanations, we may be able to give partial credit for an answer with some imprecision.

Recall the definition of the set of rational numbers,
$$Q = \{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\}.$$
We define the set of irrational numbers ,
$$\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$$

(a) Give a witness that could be used to prove the statement
$$\exists x \in \mathbb{Q} \ \forall y \in \overline{\mathbb{Q}} \ \ (x \cdot y \in \mathbb{Q})$$
Solution: The translation of this statement is "There is a rational number whose product with any irrational is rational". A witness to this statement is x = 0 . 0 is rational because we can write it as $\frac{0}{1}$ whose numerator and denominator are both integers and whose denominator is nonzero. Moreover, for an arbitrary irrational y, since y is a real number, the properties of real number multiplication give that 0 · y = 0, which (as we proved above) is rational.

(b) Give a counterexample that could be used to disprove the statement

$$\forall x \in \mathbb{Q} \quad (x \leq x^4)$$

Solution: The translation of this statement is “Every rational number is less than its 4th power”. A counterexample that can be used to disprove this statement is x = 0.5 . This is a rational number, as witnessed by the fraction $0.5 = \frac{1}{2}$ whose numerator and denominator are both integers and whose denominator is nonzero. Comparing x = 0.5 and $x^4 = 0.0016$, we see $x > x^4$ and hence $\neg(x \leq x^4)$.

(c) Give a witness that could be used to prove the statement

$$\exists (x, y) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}} \quad (x - y \notin \overline{\mathbb{Q}})$$

Solution: The translation of this statement is “There is a pair of irrational numbers (x,y) such that (x-y) is not irrational”. A witness to this statement is $(\sqrt{2}, \sqrt{2})$. Each component of this ordered pair is irrational. Computing: x−y = 0, a rational number as proved in part (a).

Problem 3 (20 points)

For each part of this problem, you **need** to justify your answer. The proof should be clear, complete, and correct: variables clearly declared, proof strategy correctly identified and applied (e.g. for induction base case and induction step each labelled) with assumptions and goals clearly articulated, calculations well supported and explained.

a) Prove or disprove the following statement:

$$\exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{Z}^{\geq n_0} (2n < n^2)$$

Solution: This statement is true. To prove it, we consider the witness $n_0 = 3$, a natural number. We need to prove the universal claim

$$\forall n \in \mathbb{Z}^{\geq 3} \quad (2n < n^2)$$

We proceed by mathematical induction:
Basis Case: Evaluating the inequality at n = 3, we get LHS = 6 and RHS = 9.
Recursive Step: Consider an arbitrary integer greater than or equal to 3 and let it be n. Assume, as the Induction Hypothesis (IH), that $2n < n^2$. We need to show that $2(n+1) < (n+1)^2$.
This follows from:
$(n+1)^2 = n^2 + 2n + 1 > 2n + 2n + 1 > 2n + 2$
where the first inequality follows from the induction hypothesis, the second from $2n > 1$ for all $n > 1$.

b) Prove or disprove the following statement:

$$\exists C \in \mathbb{Z}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{Z}^{\geq n_0}(n^2 < C(n^2 - 1))$$

Solution: This statement is true. To prove it, we consider the witnesses C = 2 (an integer),
$n_0 = 2$ (a natural number). We need to prove the universal claim:

$$\forall n \in \mathbb{Z}^{\geq 2}\ \ (\ n^2 < 2(n^2 - 1)\ )$$

Consider an arbitrary integer n greater than or equal to 2. Rewriting the required inequality
by subtracting n^2 from both sides, we need to prove that $0 < n^2 - 2$. Since $n \geq 2$, we have $n^2 - 2 \geq 2$.

Problem 4 (20 points)

(a) **Definition** A mystery function $mystery : L \to \mathbb{N}$ is defined by:

Basis Step: $mystery([]) = 0$

Recursive Step:
If $l \in L$ and $n \in \mathbb{N}$, then $\text{mystery}((n, l)) = \begin{cases} \text{mystery}(l) & \text{if mystery}(l) > n \\ n & \text{otherwise} \end{cases}$

Evaluate the function application

$$\text{mystery}((3, (0, (1, []))))$$

For full credit, include all intermediate steps with brief justifications for each.

Solution: By the recursive definition of mystery, to apply it to (3,(0,(1, []))), we need to compare 3 with mystery((0,(1, []))). Similarly, to calculate mystery((0,(1, []))), we need to compare 0 with mystery((1, [])). And, to calculate mystery((1, [])), we compare 1 with mystery([]) = 0 (which is the basis step).

Since 0 ≤ 1, mystery((1, [])) = 1.
Since 1 > 0, mystery((0,(1, []))) = 1.
Since 1 ≤ 3, mystery((3,(0,(1, [])))) = 3 .

(b) Give a precise recursive definition of the predicate sorted on the domain L which evaluates to T if the data in nodes in the linked lists are in non-increasing order and evaluates to F otherwise. For example,

$$\text{sorted}(\ (5,(2,(2,(1,[]))))\ ) = T \quad \text{sorted}(\ []\ ) = T \quad \text{sorted}(\ (2,(4,[]))\ ) = F$$

Solution: To test whether the list is sorted, we can compare the current head node with the max of the tail of the list and then recurse if we haven't found a failure of sortedness. The function mystery from part (a) computes the max data value in the list and we can use it ("as a subroutine"). Formally $\text{sorted} : L \to \{T, F\}$:

Basis Step: $\text{sorted}([]) = T$

Recursive Step:
If $l \in L$ and $n \in \mathbb{N}$, then $\text{sorted}((n,l)) = \begin{cases} F & \text{if } n < \text{mystery}(l) \\ \text{sorted}(l) & \text{otherwise} \end{cases}$

Problem 5 (20 points)

Recall that a hex color is a nonnegative integer, n, that has a base 16 fixed-width 6 expansion

$$n = (r_1 r_2 g_1 g_2 b_1 b_2)_{16,6}$$

where $(r_1 r_2)_{16,2}$ is the red component, $(g_1 g_2)_{16,2}$ is the green component, and $(b_1 b_2)_{16,2}$ is the blue. The set of all hex colors, C, is defined using set builder notation as $C = \{n \in N \mid n < 16^6\}$. We define the following sets

$NR = \{c \in C \mid c \bmod 16^4 = 0\}$
$NB = \{c \in C \mid c \text{ div } 16^2 = 0\}$

(a) Give one example of an element of $NR \times NB$. Justifications aren't required for credit for this part of the question, but it's good practice to think about how you would explain why your answer is correct.
Solution: An example is (0, 0). Any legitimate solution should receive full credit.

(b) Give three distinct examples of elements of $P(NR - \{0\})$. Justifications aren't required for credit for this part of the question, but it's good practice to think about how you would explain why your answer is correct.
Solution: Examples are (1) $\emptyset$ (2) $\{16^4\}$ (3) $NR - \{0\}$

(c) Prove or disprove the statement: $NR \subseteq NB$

Solution: This statement is false. Consider the following counterexample $x = 16^4$.

$x \in NR$

Because $x \ mod \ 16^4 = 0$. However, $x \ div \ 16^2 = 16^2$, so $x \notin NB$, thus $NR \subseteq NB$ is false.

(d) Prove or disprove the statement: $N\,R \cap N\,B \ = \ \varnothing$

Solution: This statement is false. Consider the following counterexample $x = 0$. $x \in NR$

Because $x \ mod \ 16^4 = 0$. $x \in NR$ because $x \ div \ 16^2 = 16^2$, thus $NR \bigcap NB \neq \varnothing$

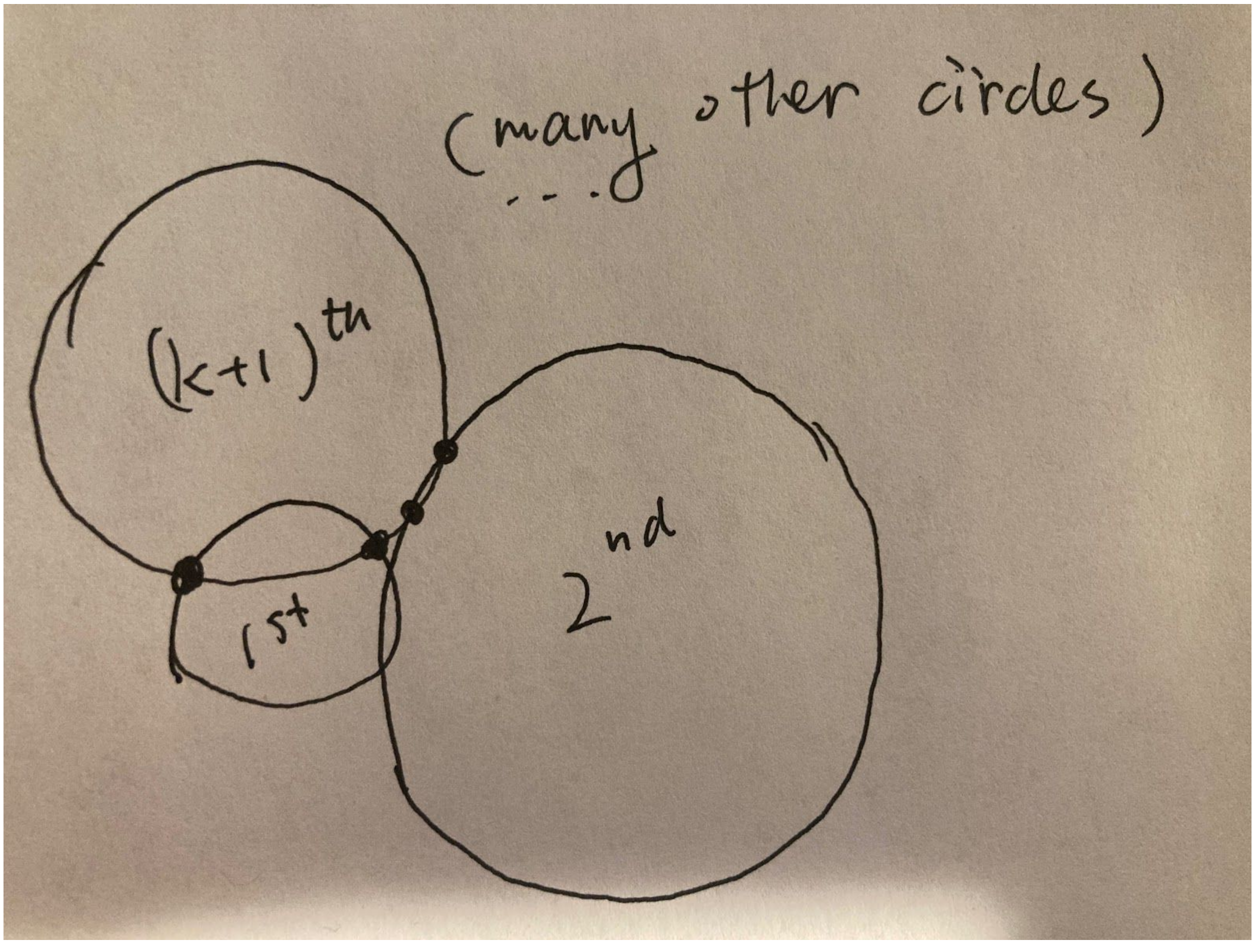
Problem 6 - Bonus (10 points)

Show that $n \ (n \geq 1)$ circles divides the plane into $n^2 - n + 2$ regions if every two circles intersect in exactly two points and no three circles contain a common point.

Solution:

When $n = 1$, obviously there're $2 = 0^2 - 0 + 2$ regions.

Suppose this claim holds when $n = k$. Consider the (k+1)-th circle. Per the assumptions stated, there're $2k$ intersection points on this circle. Since there's no three circles containing a common point, there's no other intersection point between any two intersection points that are made by the same pair of circles. A brief illustration is shown below:



These intersections divides the (k+1)-th circle into $2k$ non-overlapping arcs. Since each arc creates a new region, there're $2k$ more regions when $n = k+1$ than when $n = k$. So there're $(k^2 - k + 2) + (2k) = k^2 + k + 2 = (k+1)^2 - (k+1) + 2$ regions when $n = k+1$, which implies the claim holds when $n = k+1$. Therefore, the claim holds for all $n \geq 1$.