CSE 20, Fall 2020 - Homework 5

Due: Monday 11/16 at 11 am PDT

Instructions

Upload a single file to Gradescope for each group. All group members' names and PIDs should be on each page of the submission. You should select appropriate pages for each question when submitting to Gradescope. Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain how you arrived at your conclusions, using mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to convince the reader that your results and methods are sound.

Reading Section 1.5 Table 1 (p. 60); Section 2.1, Definitions 1-3 (pp. 116-119), Definitions 6-8 (pp. 121-123); Section 2.2 Definitions 1-5 (pp. 127-129) and Table 1 (p. 130), Section 5.3 Definition of Structural Induction

Key Concepts Proof Strategies, Set Definitions and Induction

Problem 1 (20 points)

Use mathematical induction to show that
$1^3 + 3^3 + 5^3 + ... + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$
whenever n is a positive integer.

Problem 2 (20 points)

For each part of this problem, clearly give a witness or counterexample that would be appropriate. You do not need to justify your answer. However, if you include clear explanations, we may be able to give partial credit for an answer with some imprecision.

Recall the definition of the set of rational numbers,

$$Q = \{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\}.$$

We define the set of irrational numbers ,

$$\overline{\mathbb{Q}} = \mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$$

(a) Give a witness that could be used to prove the statement

$$\exists x \in \mathbb{Q}\ \forall y \in \overline{\mathbb{Q}}\ \ (x \cdot y \in \mathbb{Q})$$

(b) Give a counterexample that could be used to disprove the statement

$$\forall x \in \mathbb{Q}\ \ (x \leq x^4)$$

(c) Give a witness that could be used to prove the statement

$$\exists (x, y) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}\ \ (x - y \notin \overline{\mathbb{Q}})$$

Problem 3 (20 points)

For each part of this problem, you **need** to justify your answer. The proof should be clear, complete, and correct: variables clearly declared, proof strategy correctly identified and applied (e.g. for induction base case and induction step each labelled) with assumptions and goals clearly articulated, calculations well supported and explained.

a) Prove or disprove the following statement:

$$\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{Z}^{\geq n_0}(2n < n^2)$$

b) Prove or disprove the following statement:

$$\exists C \in \mathbb{Z}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{Z}^{\geq n_0}(n^2 < C(n^2 - 1))$$

Problem 4 (20 points)

(a) **Definition** A mystery function $mystery : L \to \mathbb{N}$ is defined by:

Basis Step: $mystery([]) = 0$

Recursive Step:
If $l \in L$ and $n \in \mathbb{N}$, then $\text{mystery}((n, l)) = \begin{cases} \text{mystery}(l) & \text{if mystery}(l) > n \\ n & \text{otherwise} \end{cases}$

Evaluate the function application

$$\text{mystery}((3, (0, (1, []))))$$

For full credit, include all intermediate steps with brief justifications for each.

(b) Give a precise recursive definition of the predicate sorted on the domain L which evaluates to T if the data in nodes in the linked lists are in non-increasing order and evaluates to F otherwise. For example,

$\text{sorted}(\ (5, (2, (2, (1, []))))\ ) = T \quad \text{sorted}(\ []\ ) = T \quad \text{sorted}(\ (2, (4, []))\ ) = F$

Problem 5 (20 points)

Recall that a hex color is a nonnegative integer, n, that has a base 16 fixed-width 6 expansion

$$n = (r_1 r_2 g_1 g_2 b_1 b_2)_{16,6}$$

where $(r_1 r_2)_{16,2}$ is the red component, $(g_1 g_2)_{16,2}$ is the green component, and $(b_1 b_2)_{16,2}$ is the blue. The set of all hex colors, C, is defined using set builder notation as $C = \{n \in N \mid n < 16^6\}$. We define the following sets

$NR = \{c \in C \mid c \ mod \ 16^4 = 0\}$
$NB = \{c \in C \mid c \ div \ 16^2 = 0\}$

(a) Give one example of an element of $NR \times NB$. Justifications aren't required for credit for this part of the question, but it's good practice to think about how you would explain why your answer is correct.

(b) Give three distinct examples of elements of $P(NR - \{0\})$. Justifications aren't required for credit for this part of the question, but it's good practice to think about how you would explain why your answer is correct.

(c) Prove or disprove the statement: $NR \subseteq NB$

(d) Prove or disprove the statement: $N\ R \cap N\ B\ = \varnothing$

Problem 6 - Bonus (10 points)

Show that $n\ (n \geq 1)$ circles divides the plane into $n^2 - n + 2$ regions if every two circles intersect in exactly two points and no three circles contain a common point.