CSE 20 Discussion
Week 7

TA

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Show the set of prime numbers is infinite

- Use contradiction: Assume it's a finite set. Prove there must be a contradiction.
- If finite, then there exist a 'largest prime number'. We prove there exist a larger prime than that.
- Think about it: if $a = kc + d, d < c$, then is c a factor of a?

Cardinality

$|A| \leq |B|$ means there is a one-to-one function from A to B.

$$\exists f : A \to B \; \forall a_1 \in A \; \forall a_2 \in A(\; a_1 \neq a_2 \; \to \; f(a_1) \neq f(a_2) \;)$$

$|A| \geq |B|$ means there is an onto function from A to B.

$$\exists f : A \to B \; \forall b \in B \; \exists a \in A(\; f(a) = b \;)$$

$|A| = |B|$ means there is a bijection from A to B.

$$\exists f : A \to B \; \forall b \in B \; \exists a \in A(\; f(a) = b \; \wedge \; \forall a' \in A \; (a \neq a' \to f(a') \neq b) \;)$$

Cantor-Schroder-Bernstein Theorem:

$|A| = |B|$ iff $|A| \leq |B|$ and $|B| \leq |A|$ iff $|A| \geq |B|$ and $|B| \geq |A|$

HILBERT’S GRAND HOTEL Paradox

- The famous mathematician David Hilbert invented the notion of the **Grand Hotel**, which has a countably infinite number of rooms, each occupied by a guest
- When a new guest arrives…
- Can he/she be accommodated?
- Each existing guest has exactly one family member to visit him/her. These family members must be accommodated in singly-occupied rooms. Is this arrangement possible?
- What if each existing guest has N>0 family members to visit?

Show that the set of all integers is countable

- We know that positive integers are countable
- How to incorporate negative ones and zero?

Show that the set of rational numbers is countable

- Roadmap: prove the set of ‘positive rational numbers is countable’ first

- What’s the definition of rational numbers?
- We know that positive integers are countable. How to build up a mapping between positive integers and positive rational numbers?

Show that the set of positive rational numbers is countable

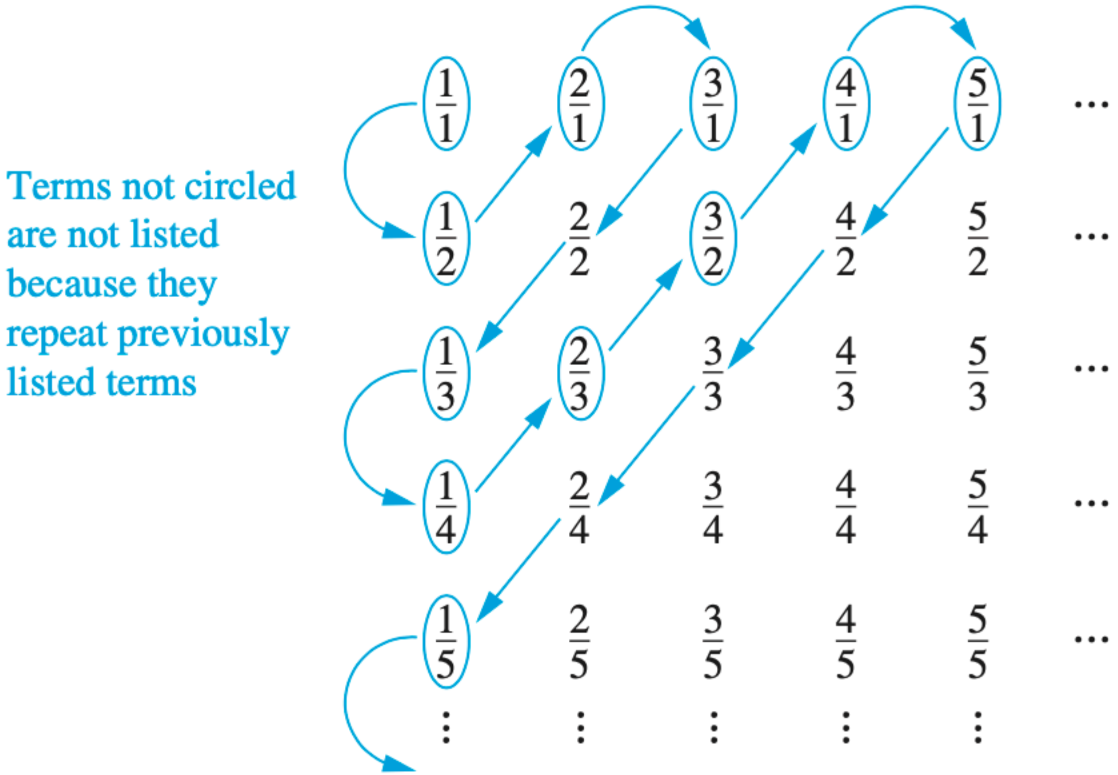

FIGURE 3 **The Positive Rational Numbers Are Countable.**

Show that the set of all rational numbers is countable

- Expanding the concept of ’positive rational numbers’ to ‘all rational numbers’
- Remember the first question we discussed?

- Twin question:
- Show $\{x \in C | x = a + bi,\ a, b \in Z\}$ is countable

Show that the set of real number is uncountable

- Intuition is important!
- We know integers are countable. What do the points representing integers look like on number axis?
- What do the points for real number look like?

- If intuition tells you real number is uncountable, you have to come up with a contradiction to prove it.
- Contradiction: Assume we can count real numbers. Then prove there exists one particular real number, it isn’t equal to any real numbers we have ever counted.

Show that the set of real number is uncountable

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}\ \ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}\ \ldots$$
$$r_3 = 0.d_{31}d_{32}d_{33}d_{34}\ \ldots$$
$$r_4 = 0.d_{41}d_{42}d_{43}d_{44}\ \ldots$$
$$\vdots$$

$r = 0.d_1d_2d_3d_4\ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 \text{ if } d_{ii} \neq 4 \\ 5 \text{ if } d_{ii} = 4. \end{cases}$$