CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Today's learning goals

- Define a predicate over a finite domain using a table of values and as properties
- Determine the truth value of the proposition resulting from evaluating a predicate
- Describe the set of domain elements that make a predicate with one input evaluate to true.
- Evaluate universal and existential statements about finite domains (with no quantifier alternations).
- Counterexample and witness-based arguments for predicates with infinite domains
- Practice combinations of $\wedge$, $\rightarrow$ in conjunction with universal and existential quantifiers
- State and apply DeMorgan’s law for quantified statements.

Predicates as tables

A **predicate** is a function from a given set (domain) to {T,F}
It can be specified by its input-output definition table.
It can be applied, or **evaluated at**, an element of the domain

x	$P(x)$ $[x]_{2c,3} > 0$
000	F
001	T
010	T
011	T
100	F
101	F
110	F
111	F

Domain is

P(001) is

Predicates as functions

A **predicate** is a function from a given set (domain) to {T,F}
It can be specified by its input-output definition table **or by specifying the rule**
It can be applied, or **evaluated at**, an element of the domain.

x	$P(x)$ $[x]_{2c,3} > 0$	$N(x)$ $[x]_{2c,3} < 0$
000	F	
001	T	
010	T	
011	T	
100	F	
101	F	
110	F	
111	F	

Predicates as functions

x	$Mystery(x)$
000	T
001	T
010	T
011	F
100	F
101	T
110	F
111	T

Which of the following is a description of the rule that corresponds to the input-output definition table for *Mystery(x)* ?

A: "$[x]_{2c,3}$ is a non-zero number"
B: "$[x]_{2c,3}$ is a non-negative number"
C: "$[x]_{2c,3}$ is a number that is even and positive"
D: "$[x]_{2c,3}$ is a number that is not positive and is not negative"
E: None of the above

Predicates as functions

A **predicate** is a function from a given set (domain) to {T,F}
It can be specified by its input-output definition table or by specifying the rule
or by specifying its truth set: the elements of the domain at which the predicate evaluates to T

Input	Output		
	$P(x)$	$N(x)$	$Mystery(x)$
x	$[x]_{2c,3} > 0$	$[x]_{2c,3} < 0$	
000	F		T
001	T		T
010	T		T
011	T		F
100	F		F
101	F		T
110	F		F
111	F		T

Truth set for P(x) is

Truth set for N(x) is

Truth set for *Mystery(x)* is

Why is specifying the truth set of a predicate enough to define its rule?

Quantified statements

Rosen p. 40-45

We can make claims about a set by saying which or how many of its elements satisfy a property. These claims are called quantified statements and use predicates.

The **universal quantification** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain" and is written $\forall x P(x)$.

The **existential quantification** of $P(x)$ is the statement "There exists an element x in the domain such that $P(x)$" and is written $\exists x P(x)$.

The **universal quantification** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain" and is written $\forall x P(x)$.

Which of the following is a true statement?

A. $\forall x \; P(x)$

B. $\forall x \; N(x)$

C. $\forall x \; Mystery(x)$

D. All of the above

E. None of the above

Input	Output		
	$P(x)$	$N(x)$	$Mystery(x)$
x	$[x]_{2c,3} > 0$	$[x]_{2c,3} < 0$	
000	F		T
001	T		T
010	T		T
011	T		F
100	F		F
101	F		T
110	F		F
111	F		T

What about the existential quantification of each of these predicates?

Counterexamples and Witnesses

Definition: An element for which $P(x)$ is F is called a **counterexample** of $\forall x P(x)$.

Definition: An element for which $P(x)$ is T is called a **witness** of $\exists x P(x)$.

For a predicate E(x), which of the following is a valid proof strategy:

A: We can prove that $\forall x E(x)$ is true using a witness.
B: We can prove that $\forall x E(x)$ is true using a counterexample.
C: We can prove that $\forall x E(x)$ is false using a witness.
D: We can prove that $\forall x E(x)$ is false using a counterexample
E: More than one of the above

Counterexamples and Witnesses

Input	Output		
	$P(x)$	$N(x)$	$Mystery(x)$
x	$[x]_{2c,3} > 0$	$[x]_{2c,3} < 0$	
000	F		T
001	T		T
010	T		T
011	T		F
100	F		F
101	F		T
110	F		F
111	F		T

Which of these is true?

A: $\forall x(\ P(x) \lor N(x)\ )$

B: $\exists x(\ P(x) \to N(x)\ )$

C: $\forall x(\ P(x) \oplus Mystery(x)\ )$

D: More than one of the above

E: None of the above

Quantifier De Morgan

Rosen p. 45

Quantifier version of De Morgan's laws: $\boxed{\neg \forall x P(x) \equiv \exists x\, (\neg P(x))}$ $\boxed{\neg \exists x Q(x) \equiv \forall x\, (\neg Q(x))}$

Example: ________________ is a false universal quantification. It is logically equivalent to ________________

Recall: Each RNA strand is a string whose symbols are elements of the set $B = \{\texttt{A}, \texttt{C}, \texttt{G}, \texttt{U}\}$. The **set of all RNA strands** is called S. The function *rnalen* that computes the length of RNA strands in S is:

$$\begin{array}{lll} & & rnalen : S \to \mathbb{Z}^+ \\ \text{Basis Step:} & \text{If } b \in B \text{ then} & rnalen(b) = 1 \\ \text{Recursive Step:} & \text{If } s \in S \text{ and } b \in B\text{, then} & rnalen(sb) = 1 + rnalen(s) \end{array}$$

Example predicates on S

$H(s) = T$	Truth set of H is ______________
$L_3(s) = \begin{cases} T & \text{if } rnalen(s) = 3 \\ F & \text{otherwise} \end{cases}$	Strand where L_3 evaluates to T is e.g.________ Strand where L_3 evaluates to F is e.g. ________
$F_{\texttt{A}}$ is defined recursively by: Basis step: $F_{\texttt{A}}(\texttt{A}) = T$, $F_{\texttt{A}}(\texttt{C}) = F_{\texttt{A}}(\texttt{G}) = F_{\texttt{A}}(\texttt{U}) = F$ Recursive step: If $s \in S$ and $b \in B$, then $F_{\texttt{A}}(sb) = F_{\texttt{A}}(s)$	Strand where $F_{\texttt{A}}$ evaluates to T is e.g.________ Strand where $F_{\texttt{A}}$ evaluates to F is e.g. ________
$P_{\texttt{AUC}}$ is defined as the predicate whose truth set is the collection of RNA strands where the string AUC is a substring (appears inside s, in order and consecutively)	Strand where $P_{\texttt{AUC}}$ evaluates to T is e.g.________ Strand where $P_{\texttt{AUC}}$ evaluates to F is e.g. ________

$H(s) = T$
$L_3(s) = \begin{cases} T & \text{if } rnalen(s) = 3 \\ F & \text{otherwise} \end{cases}$
$F_{\texttt{A}}$ is defined recursively by: Basis step: $F_{\texttt{A}}(\texttt{A}) = T$, $F_{\texttt{A}}(\texttt{C}) = F_{\texttt{A}}(\texttt{G}) = F_{\texttt{A}}(\texttt{U}) = F$ Recursive step: If $s \in S$ and $b \in B$, then $F_{\texttt{A}}(sb) = F_{\texttt{A}}(s)$
$P_{\texttt{AUC}}$ is defined as the predicate whose truth set is the collection of RNA strands where the string AUC is a substring (appears inside s, in order and consecutively)

A true universal quantification is:

A false universal quantification is:

A true existential quantification is:

A false existential quantification is:

For next time

- Read website carefully

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- Next pre-class reading:
  - Section 2.1 Definitions 8 and 9 (p. 123). Section 1.5 Example 1 (p. 57) and Example 4 (p. 59)