CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

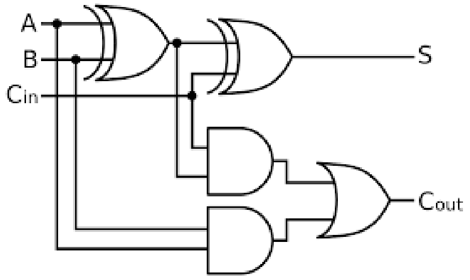
Today's learning goals

- Translate sentences from English to propositional logic using appropriate propositional variables and boolean operators.
- Evaluate the truth value of a compound proposition given truth values of its constituent variables.
- Prove propositional equivalences using truth tables
- Prove propositional equivalences using other known equivalences, e.g.
  - DeMorgan’s laws
  - Double negation laws
  - Distributive laws, etc.
- Compute the CNF and DNF of a given compound proposition.

Logic

Rosen Section 1.1

- Use gates and circuits to express arithmetic.
  - Write desired input-output table
  - Translate to gates & wires



- Precisely express true facts and invariant statements.

- Identify valid arguments (patterns of reasoning) that could be used in proofs.

Definitions

Rosen pp. 2-4

- **Proposition**: declarative sentence that is T or F (not both)
- **Propositional variable**: variables that represent propositions.
- **Compound proposition**: new propositions formed from existing propositions using logical operators.
- **Truth table**: table with 1 row for each of the possible combinations of truth values of the input and an additional column that shows the truth value of the result of the operation corresponding to a particular row.

Logical operators aka propositional connectives

Input		Output
p	q	p ^ q
T	T	T
T	F	F
F	T	F
F	F	F

“Both p and q are true”

Conjunction

Input		Output
p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

“Exactly one of p and q is true”

Exclusive or

Input		Output
p	q	p v q
T	T	
T	F	
F	T	
F	F	

“At least one of p and q is true”

Disjunction

Fill in the output for disjunction (reading top to bottom):

A. T-T-T-F
B. T-F-T-F
C. F-F-F-T
D. T-F-F-T
E. None of the above

Logical operators aka propositional connectives

Input		Output
p	q	p ^ q
T	T	T
T	F	F
F	T	F
F	F	F

“Both p and q are true”

Conjunction

Input		Output
p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

“Exactly one of p and q is true”

Exclusive or

Input		Output
p	q	p v q
T	T	T
T	F	T
F	T	T
F	F	F

“At least one of p and q is true”

Disjunction

Input	Output
p	$\neg p$
T	F
F	T

Negation

Truth tables

Rosen p. 10

We can use truth tables to compute value of compound proposition.

Input			Output	
p	q	r		$(p \wedge q) \oplus ((p \oplus q) \wedge r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Logical Equivalence

Rosen p. 25

We can use truth tables to compute value of compound proposition.

Input			Output	
p	q	r	$(p \wedge q) \oplus ((p \oplus q) \wedge r)$	$(p \wedge q) \vee ((p \oplus q) \wedge r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Compound propositions that have the same truth values for all settings of truth values to their propositional variables are **logically equivalent**, denoted $\equiv$

Tautology and contradiction

Rosen p. 25

Tautology: compound proposition that evaluates to true for all settings of truth values to its propositional variables; it is abbreviated T.

Contradiction: compound proposition that evaluates to false for all settings of truth values to its propositional variables; it is abbreviated F.

Which of the following is a tautology?

A. $p \wedge p$
B. $p \oplus p$
C. $p \vee p$
D. $p \vee \neg p$
E. $p \wedge \neg p$

Which (if any) is a contradiction?

(Some) logical equivalences

Rosen p. 26-28

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$p \wedge F \equiv F$$

$$p \wedge T \equiv p$$

$$p \vee T \equiv T$$

$$p \vee F \equiv p$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

…. 32 equivalences listed in book!

Can replace *p* and *q* with any (compound) proposition

Going backwards

Given a compound proposition, we can use

- Truth tables
- Logical equivalences

to compute its truth value for specific input values.

What about the opposite problem? Given truth table settings, want a compound proposition with that output.

- Apply to design a circuit

Reverse-engineering

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

Algorithmic approach 1

Which situations guarantee output T?

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

Algorithmic approach 1

Which situations guarantee output T?

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

ONLY THIS ROW for *mystery*$_2$

Algorithmic approach 1

Which situations guarantee output T?

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

ONLY THIS ROW for *mystery*$_2$

"p is False and q is False"

$\neg p \wedge \neg q$

Algorithmic approach 1

Which situations guarantee output T?

Input		Outp
p	q	$mystery_1$
T	T	T
T	F	T
F	T	F
F	F	T

Which compound proposition gives output *mystery*$_1$?

A. $\neg p \wedge q$

B. $(p \wedge q) \wedge (p \wedge \neg q) \wedge (\neg p \wedge \neg q)$

C. $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

D. More than one of the above

E. None of the above

Algorithmic approach 2

Which situations avoid output F?

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

AVOID ONLY THIS ROW for $mystery_1$

Algorithmic approach 2

Which situations avoid output F?

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

AVOID ONLY THIS ROW for $mystery_1$

$$\neg(\neg p \wedge q)$$

Algorithmic approach 2

Which situations avoid output F?

Input		Output	
p	q	$mystery_1$	$mystery_2$
T	T	T	F
T	F	T	F
F	T	F	F
F	F	T	T

AVOID ONLY THIS ROW for *mystery*$_1$

$$\neg(\neg p \wedge q)$$

i.e. $p \vee \neg q$

DNF and CNF

Rosen p. 35 #42-53

Disjunctive normal form: OR of ANDs (of variables or their negations).

For *mystery*$_1$ a DNF is $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

For *mystery*$_2$ a DNF is $\neg p \wedge \neg q$

Conjunctive normal form: AND of ORs (of variables or their negations).

For *mystery*$_1$ a CNF is $p \vee \neg q$

For *mystery*$_2$ a CNF is $(\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee \neg q)$

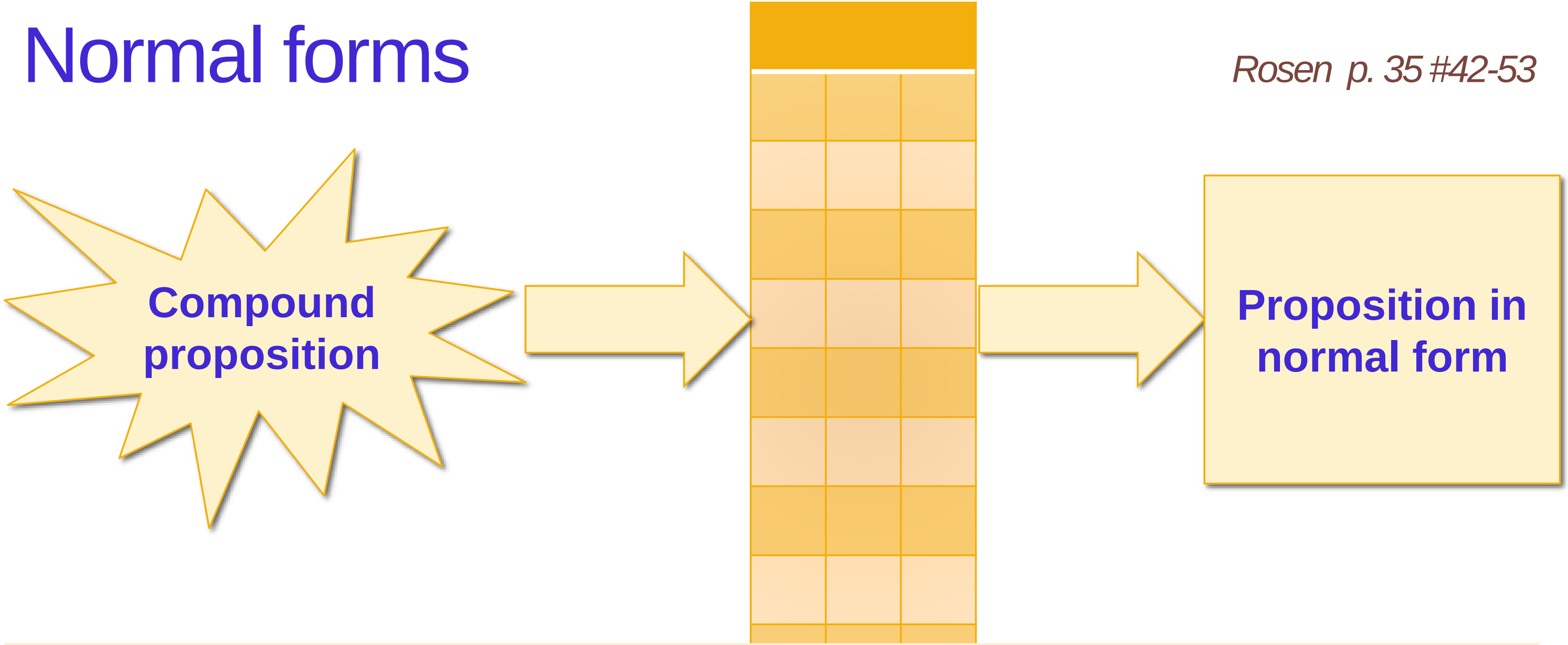
Payoff

- Any output column of a truth table (assignment of T/F to each setting of truth values to its propositional variables) can be realized as a compound proposition.

- Given any input-output relationship, we can build a circuit implementing it using CNF / DNF

Normal forms

Rosen p. 35 #42-53



Added benefit: If want to reduce connectives further to prove a new collection of connectives is functionally complete, only need to consider those used in normal form.

REVERSE ENGINEERING A FORMULA

(IF WE HAVE TIME)

Reverse-engineering

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Algorithmic approach 1

Which situations guarantee output T?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Algorithmic approach 1

Which situations guarantee output T?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

ANY ONE OF THESE ROWS!

Generalizing AND and OR

Rosen p. 28

Associativity $(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

So we can "drop" the parentheses when using the **same** operation multiple times

$p \vee q \vee r$ $p \wedge q \wedge r$

Algorithmic approach 1

Which situations guarantee output T?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

$p \wedge q \wedge r$

$p \wedge q \wedge \neg r$

$p \wedge \neg q \wedge \neg r$

$\neg p \wedge \neg q \wedge r$

Algorithmic approach 1

Which situations guarantee output T?

p	q	r	?	
T	T	T	T	$p \wedge q \wedge r$
T	T	F	T	$p \wedge q \wedge \neg r$
T	F	T	F	
T	F	F	T	$p \wedge \neg q \wedge \neg r$
F	T	T	F	
F	T	F	F	
F	F	T	T	$\neg p \wedge \neg q \wedge r$
F	F	F	F	

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

Algorithmic approach 2

Which situations avoid output F?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Algorithmic approach 2

Which situations avoid output F?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

AVOID ALL OF THESE ROWS!

Algorithmic approach 2

Which situations avoid output F?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

AVOID ALL OF THESE ROWS!

$\neg(\neg p \wedge \neg q \wedge \neg r) \equiv$

$p \vee q \vee r$

Algorithmic approach 2

Which situations avoid output F?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

$\neg p \vee q \vee \neg r$

$p \vee \neg q \vee \neg r$

$p \vee \neg q \vee r$

$p \vee q \vee r$

Algorithmic approach 2

Which situations avoid output F?

p	q	r	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

$\neg p \vee q \vee \neg r$

$p \vee \neg q \vee \neg r$

$p \vee \neg q \vee r$

$p \vee q \vee r$

$$(\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$$

For next time

- Read website carefully

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

- Next pre-class reading:
  - Section 1.1 Definition 5, Definition 6, Example 11