**Claim**: $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$

**Proof**: Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

**We want to prove that** $f$ is not onto.

Rewriting using the definition of onto:

$$\neg \ ( \ \forall B \in \mathcal{P}(\mathbb{N}) \ \exists a \in \mathbb{N} \ ( \ f(a) = B \ ) \ ) \ .$$

By logical equivalence, we can write this as an existential statement:

$$\exists B \in \mathcal{P}(\mathbb{N}) \ \forall a \in \mathbb{N} \ ( \ f(a) \neq B \ )$$

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{ n \in \mathbb{N} \ \mid \ n \notin f(n) \} \, .$$

By definition of power set, since all elements of $D_f$ are in $\mathbb{N}$, it follows that

$$D_f \in \mathcal{P}(\mathbb{N}) \, .$$

Therefore, it's enough to prove the following lemma:

**Lemma**: $\forall a \in \mathbb{N} \ ( \ f(a) \neq D_f \ )$.
**Proof of lemma**: Towards universal generalization, consider an arbitrary $a \in \mathbb{N}$. By definition of set equality, **we want to prove** that $\exists x (\neg (x \in f(a) \ \leftrightarrow \ x \in D_f))$. For a witness, consider $x = a$. There are two cases: $a \in f(a) \ \vee \ a \notin f(a)$. By definition of $D_f$, each guarantees that $f(a) \neq D_f$.

By the Lemma, we have proved that $f$ is not onto, and since $f$ was arbitrary, there are no onto functions from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$.

**Claim: Generalization of De-Morgan's Law to union of $n$ sets**: Show that if $A_1, A_2, \ldots\ldots, A_n$ are sets, then:

$$\overline{(A_1 \cup A_2 \ldots. A_n)} = \overline{A_1} \cap \overline{A_2} \ldots.\cap \overline{A_n}$$

You may use the basic De-Morgan's law: $\overline{(A_1 \cup A_2)} = \overline{A_1} \cap \overline{A_2}$,without proof.

**Proof:** Towards a proof by mathematical induction:

**Base Case:** The base case is for $n = 1$: $\overline{\boldsymbol{A_1}} = \overline{\boldsymbol{A_1}}$ (since the union and intersection of just one set is the set itself).

**Inductive Hypothesis:** We assume that our claim is valid for an integer $n$:

$$\overline{(A_1 \cup A_2 \ldots. A_n)} = \overline{A_1} \cap \overline{A_2} \ldots.\cap \overline{A_n}$$

**We want to show** that our claim will also be valid for $n + 1$:

$$\overline{(A_1 \cup A_2 \ldots. A_n \cup A_{n+1})} = \overline{A_1} \cap \overline{A_2} \ldots.\cap \overline{A_n} \cap \overline{A_{n+1}}$$

To prove this, we will consider the left-hand side of my claim:

$$\overline{(A_1 \cup A_2 \ldots. A_n \cup A_{n+1})} = \overline{((A_1 \cup A_2 \ldots. A_n) \cup A_{n+1})}$$

Now, I can represent the set formed by the union of $A_1$ as:
$A_1 \cup A_2 \cup \ldots\ldots A_n = B$,
where $B$ is another set.
Therefore, $\overline{((A_1 \cup A_2 \ldots. A_n) \cup A_{n+1})} = \overline{B \cup A_{n+1}}$

Applying De-Morgan's Law, we get:

$$\overline{B \cup A_{n+1}} = \overline{B} \cap \overline{A_{n+1}}$$

Therefore,

$$\overline{((A_1 \cup A_2 \ldots. A_n) \cup A_{n+1})} = \overline{((A_1 \cup A_2 \ldots. A_n)} \cap \overline{A_{n+1}}$$

From the inductive hypothesis, we know that: .

$$\overline{(A_1 \cup A_2 \ldots. A_n)} = \overline{A_1} \cap \overline{A_2} \ldots.\cap \overline{A_n}$$

Therefore,

$$\overline{((A_1 \cup A_2 \ldots. A_n) \cup A_{n+1})} = \overline{B} \cap \overline{A_{n+1}} = \overline{(A_1 \cup A_2 \ldots. A_n)} \cap \overline{A_{n+1}}$$

$$\overline{(A_1 \cup A_2 \ldots. A_n)} \cap \overline{A_{n+1}} = \overline{A_1} \cap \overline{A_2} \ldots.\cap \overline{A_n}$$

$$\boldsymbol{Q.E.D}$$

**Lemma:** Use mathematical induction to show that given a set **A** of $n + 1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

**Proof:** Towards a proof by induction:

**Base Step:** For $n = 1$, we have a set of 2 positive integers, all of which are less than $2n = 2$. Therefore, the possible sets are: $\{1,1\}, \{1,2\}, \{2,1\}, \{2,2\}$. We observe that in each of these sets, there is one element that perfectly divides the other.

**Induction Hypothesis:** We assume that for a set of $n + 1$ positive integers, there is at least one element in the set that divides another integer in the set.

**We want to show** that our claim will also be valid for $n + 2$: there is at least one element in the set, less than or equal to $2n + 2$ , that divides another integer in the set.

Each element of the set is written as: $x_i$ (representing the $ith$ element of the set).

Again, we will deal with this proof by defining multiple cases:

**Case 1:** There are at least $n + 1$ elements in the set $A$ that are less than or equal to $2n$. In this case, these $n + 1$ elements can be written down as a separate set $U_{n+1}$ and the last element (which may or may not be greater than $2n$) is $x_{n+2}$.

$$\boldsymbol{A = U_{n+1} \cup \{x_{n+2}\}}$$

However, by the induction hypothesis, we know that $U_{n+1}$ is a set of $n + 1$ elements, all less than $2n$. Therefore, there must exist at least one integer here that divides another. Since $A$ has all the elements of $U_{n+1}$, this condition will hold for A as well.

**Case 2:** There are 2 elements in the set $A$ that are greater than $2n$. There can be sub-cases here. If both these elements are equal ($2n + 1$ or $2n + 2$), then the lemma is trivially satisfied. However, if the 2 elements are $2n + 1$ and $2n + 2$, then we cannot comment in general about the divisibility of the rest of the elements directly. In this case, if my set has 2, then $(2n + 2)$ will be directly divisible by 2, trivially.
However, if we don't have a 2, then we will need to use a trick here. To use induction, I need to reduce this statement to the form given for $n + 1$ integers. To do this, I will take $2n + 2$ and replace this with $n + 1$ in my set. Why does this work? Because $2n + 2$ cannot possibly divide any other integer, and the only integer $n + 1$ can divide is $2n + 2$, which I have removed from my set. Therefore, I am not changing the divisibility

condition (instead, if my set had a 2, then this would have been trivially true) at all. However, we notice now that with $2n + 2$ removed, my set has $n + 1$ integers, all of which are less than or equal to $2n$, and I can now use the induction hypothesis here to claim that in this case, there still exists an integer which divides another integer.

**Case 3:** There are more than 2 elements in the set that are greater than $2n$: In this case, we must have repeated elements (since there are only 2 elements greater than $2n$: $2n + 1$ and $2n + 2$. Hence, the condition is trivially true.

Therefore, if the inductive hypothesis holds, then the proof holds!

**Lemma: Let $x \in Z$. If $x^2 - 6x + 5$ is even, then x is odd.**

**Proof:** Towards a proof by contraposition:

Suppose $x$ is even. Then we need to show that $x^2 - 6x + 5$ must be odd.
We can write an even number as: $x = 2k$
Therefore: $x^2 - 6x + 5 = (2k)^2 - 6(2k) + 5 = 4k^2 - 12k + 5$
$= 4(k^2 - 3) + 5$
Since $k$ is an integer, $k^2 - 3$ must also be an integer. Therefore, $4(k^2 - 3)$ must be even, and hence $4(k^2 - 3) + 5$ must be odd.

Hence proved!