A **finite set** is a set that is the empty set or is the same size as $\{1, \ldots, n\}$ for some nonnegative integer n.
A **countably infinite set** is a set that is the same size as $\mathbb{Z}^+$ or $\mathbb{N}$.
An **uncountable set** is a set that is not finite and is not countably infinite.

Comparing $\mathbb{Q}$ and $\mathbb{R}$ *Circle correct selection for each property.*

Finite?	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$	Both sets
Countably infinite?	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$	Both sets
Uncountable?	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$	Both sets
Has greatest element?	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$	Both sets
Has least element?	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$	Both sets
$\forall x \forall y (x < y \to \exists z (x < z < y))$	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$	Both sets
Least upper bound?	Neither set	Only $\mathbb{Q}$	Only $\mathbb{R}$?	Both sets

The set of real numbers

Order axioms (Rosen Appendix 1):

Reflexivity	$\forall a \in \mathbb{R} (a \leq a)$
Antisymmetry	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ (\ (a \leq b\ \wedge\ b \leq a) \to (a = b)\ )$
Transitivity	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ (\ (a \leq b \wedge b \leq c)\ \to\ (a \leq c)\ )$
Trichotomy	$\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ (\ (a = b\ \vee\ b > a\ \vee\ a < b)$

Completeness axioms (Rosen Appendix 1):

Least upper bound	Every nonempty set of real numbers that is bounded above has a least upper bound
Nested intervals	For each sequence of intervals $[a_n, b_n]$ where, for each n, $a_n < a_{n+1} < b_{n+1} < b_n$, there is at least one real number x such that, for all n, $a_n \leq x \leq b_n$.

Each real number $r \in \mathbb{R}$ is described by a function to give better and better approximations

$$x_r : \mathbb{Z}^+ \to \{0,1\} \qquad \text{where } x_r(n) = n^{th} \text{ bit in binary expansion of } r$$

r	Binary expansion	x_r
0.1	$0.00011001\ldots$	$x_{0.1}(n) = \begin{cases} 0 & \text{if } n > 1 \text{ and } (n \textbf{ mod } 4) = 2 \\ 0 & \text{if } n = 1 \text{ or if } n > 1 \text{ and } (n \textbf{ mod } 4) = 3 \\ 1 & \text{if } n > 1 \text{ and } (n \textbf{ mod } 4) = 0 \\ 1 & \text{if } n > 1 \text{ and } (n \textbf{ mod } 4) = 1 \end{cases}$
$\sqrt{2} - 1 = 0.4142135\ldots$	$0.01101010\ldots$	Use linear approximations (tangent lines from calculus) to get algorithm for bounding error of successive operations. Define $x_{\sqrt{2}-1}(n)$ to be n^{th} bit in approximation that has error less than $2^{-(n+1)}$.

Claim: $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is uncountable.

Approach 1: Mimic proof that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Proof: By definition of countable, since $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not finite, **to show** is $|\mathbb{N}| \neq |\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}|$.

To show is $\forall f : \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ (f is not a bijection) . Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{Z}^+ \to \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$. **To show**: f is not a bijection. It's enough to show that f is not onto. Rewriting using the definition of onto, **to show**:

$$\exists x \in \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\} \; \forall a \in \mathbb{N} \; (\; f(a) \neq x \;)$$

In search of a witness, define the following real number by defining its binary expansion

$$d_f = 0.b_1 b_2 b_3 \cdots$$

where $b_i = 1 - b_{ii}$ where b_{jk} is the coefficient of 2^{-k} in the binary expansion of $f(j)$. Since* $d_f \neq f(a)$ for any positive integer a, f is not onto.

Approach 2: Nested closed interval property

To show $f : \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not onto. **Strategy**: Build a sequence of nested closed intervals that each avoid some $f(n)$. Then the real number that is in all of the intervals can't be $f(n)$ for any n. Hence, f is not onto.

Consider the function $f : \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ with $f(n) = \frac{1+\sin(n)}{2}$

n	$f(n)$	Interval that avoids $f(n)$
0	0.5	
1	$0.920735\ldots$	
2	$0.954649\ldots$	
3	$0.570560\ldots$	
4	$0.121599\ldots$	
$\vdots$		

*There's a subtle imprecision in this part of the proof as presented, but it can be fixed.