CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Today's learning goals

- Compare sizes of sets using one-to-one, onto, and invertible functions.
- Classify sets by cardinality into: **Finite sets, countable sets, uncountable sets.**
- Explain the central idea in Cantor's diagonalization argument.

|A| ≤ |B| means there is a one-to-one function from A to B.

$$\exists f : A \to B\ \forall a_1 \in A\ \forall a_2 \in A(\ a_1 \neq a_2\ \to\ f(a_1) \neq f(a_2)\ )$$

|A| ≥ |B| means there is an onto function from A to B.

$$\exists f : A \to B\ \forall b \in B\ \exists a \in A(\ f(a) = b\ )$$

|A| = |B| means there is a bijection from A to B.

$$\exists f : A \to B\ \forall b \in B\ \exists a \in A(\ f(a) = b\ \wedge\ \forall a' \in A\ (a \neq a' \to f(a') \neq b)\ )$$

Cantor-Schroder-Bernstein Theorem:

|A| = |B| iff |A| ≤ |B| and |B| ≤ |A| iff |A| ≥ |B| and |B| ≥ |A|

Useful Lemmas … how would you prove each one?

- If A and B are countable sets, then A U B is countable.

  Theorem 1, p. 174

- If A and B are countable sets, then A x B is countable.

  Generalize pairing idea

- If A is a subset of B, to show that |A| = |B|, it's enough to give a 1-1 function from B to A or an onto function from A to B.

  Exercise 22, p. 176

- If A is a subset of a countable set, then it's countable.

  Exercise 16, p. 176

- If A is a superset of an uncountable set, then it's uncountable.

  Exercise 15, p. 176

Cardinality

Rosen Def 3 p. 171

Finite sets

|A| = |{1,...,n}| for some nonnegative int n

e.g. {1,2,3} {A, U, G, C} $\emptyset$

Countably infinite sets

$|A| = |\mathbf{Z}^+|$ or $|A| = |\mathbf{N}|$ (e.g. can be listed out)

e.g. $\mathbb{Z}^+, \mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathrm{N}, \mathbb{Z} \times \mathbb{Z}, \mathbb{Q}^+, \mathbb{Q}, L, S$

Uncountable sets

e.g. $\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathbb{Z}^+), \mathcal{P}(\mathbb{Z})$

Last lecture - diagonalization argument

Q vs. R

$\mathbb{Q}$	The set of rational numbers	$\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\}$
$\mathbb{R}$	The set of real numbers	

Which property is true of one of these sets and not the other?

A. $\forall x \exists y (x < y)$

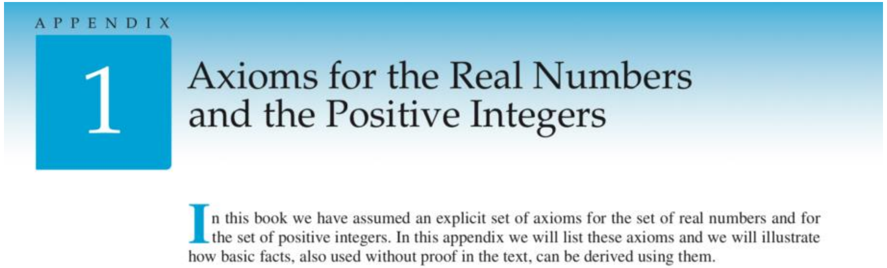
B. $\forall x \exists y (y < x)$

C. $\forall x \forall y (x < y \rightarrow \exists z (x < z < y))$

D. All of the above

E. None of the above

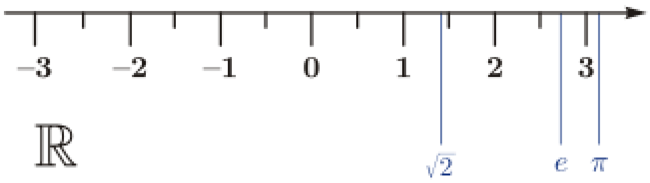
$\mathbb{Z}$	The set of integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers	$\{1, 2, \ldots\}$
$\mathbb{N}$	The set of nonnegative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers	



APPENDIX 1

Axioms for the Real Numbers and the Positive Integers

In this book we have assumed an explicit set of axioms for the set of real numbers and for the set of positive integers. In this appendix we will list these axioms and we will illustrate how basic facts, also used without proof in the text, can be derived using them.



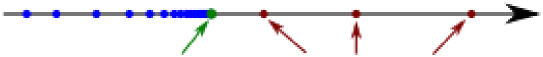
"Every real number has a unique decimal expansion (when the possibility that the expansion that has a tail end that consists entirely of the digit 9 is excluded)". Rosen p174

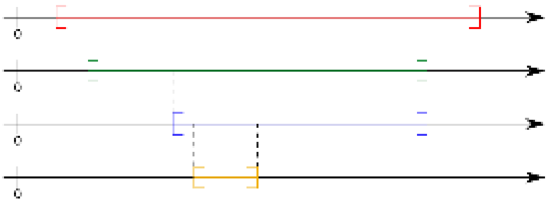
"Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced." Wikipedia

$\mathbb{Z}$	The set of integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers	$\{1, 2, \ldots\}$
$\mathbb{N}$	The set of nonnegative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers	

Math approach: the set of real numbers

* is a superset of $\mathbb{Z}$
* is totally ordered (order axioms on worksheet)
* is **complete** / has the **nested closed interval property** (axioms on worksheet)

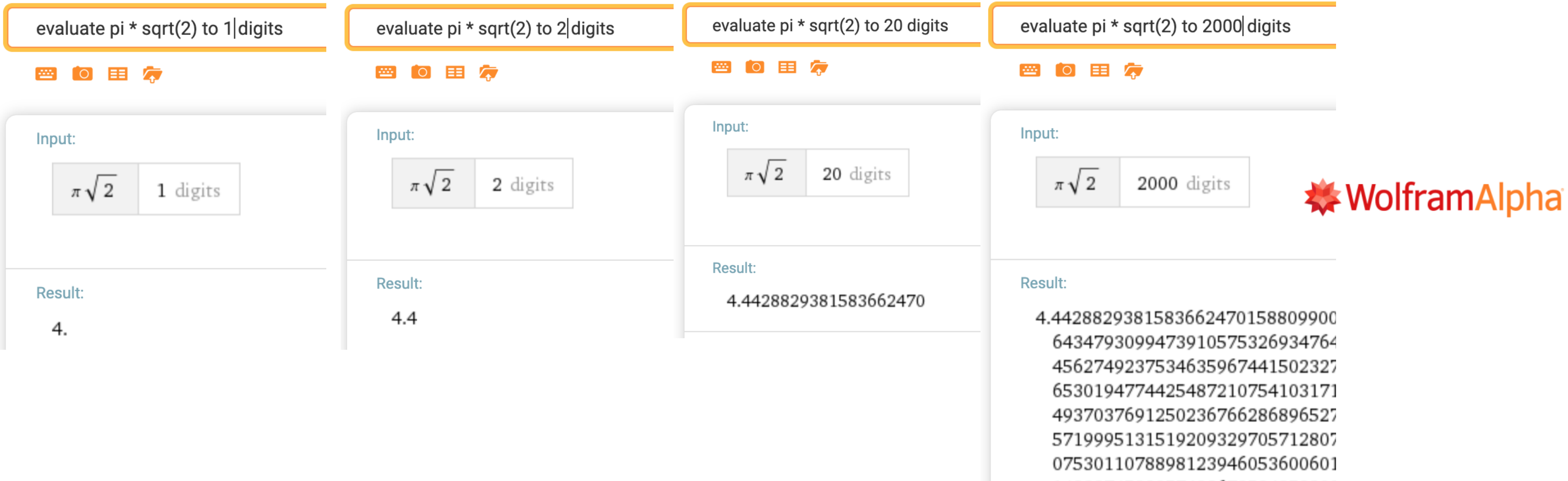




For more on this, see Math 140

CS approach: approximate real numbers; a real number between 0 and 1 is specified as a function to get better and better approximations

static double	**PI**The double value that is closer than any other to *pi*, the ratio of the circumference of a circle to its diameter.



CS approach: approximate real numbers; a real number between 0 and 1 is specified as a function to get better and better approximations

$$x_r : \mathbb{Z}^+ \to \{0,1\} \text{ where } x_r(n) = n^{th} \text{ bit in binary expansion of } r$$

r	Binary expansion	x_r
0.5	$0.10000\ldots$	$x_{0.5}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$
0.1	$0.0001100110011\ldots$	$x_{0.1}(n) = \begin{cases} 0 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \text{ and } (n \bmod 4) = 2 \\ 0 & \text{if } n > 1 \text{ and } (n \bmod 4) = 3 \\ 1 & \text{if } n > 1 \text{ and } (n \bmod 4) = 0 \\ 1 & \text{if } n > 1 \text{ and } (n \bmod 4) = 1 \end{cases}$
$\sqrt{2} - 1 = 0.41421356237\ldots$	$0.01101010000010\ldots$	Use linear approximations (tangent lines from calculus) to get algorithm for bounding error of successive operations. Define $x_{\sqrt{2}-1}(n)$ to be n^{th} bit in approximation that has error less than $2^{-(n+1)}$.

Claim: $\{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is uncountable.

Do you believe this?

A. No, this is a finite set so it's countable.
B. No, 0 and 1 are integers so this is a subset of integers so it's countable.
C. No, the absolute value of each of the numbers in the set is bounded by 1.
D. Yes, there are infinitely many numbers between 0 and 1 so the set must be uncountable.
E. Yes, the statement is called a claim so it must be true.

Note: because this is a subset of **R**, it being uncountable would guarantee that **R** is also uncountable.

Approach 1: mimic power set proof from last lecture

Example 5 Rosen p. 173

Theorem: The set $\{x \in \mathbb{R} \mid 0 < x < 1\}$ **is uncountable**

Proof: Consider an arbitrary function $f : \mathbb{Z}^+ \rightarrow \{r \in \mathbb{R} \mid 0 \leq r \land r \leq 1\}$

List its images

$f(1) = r_1 = 0.\, b_{11}\, b_{12}\, b_{13}\, b_{14} \ldots$

$f(2) = r_2 = 0.\, b_{21}\, b_{22}\, b_{23}\, b_{24} \ldots$

$f(3) = r_3 = 0.\, b_{31}\, b_{32}\, b_{33}\, b_{34} \ldots$

$f(4) = r_4 = 0.\, b_{41}\, b_{42}\, b_{43}\, b_{44} \ldots$

We're going to find a real number **d** between 0 and 1 (i.e. in the codomain of f) that is not in this list

$$d_f = 0.\, b_1\, b_2\, b_3\, b_4 \ldots$$

where $\mathbf{b_i = 1 - b_{ii}}$

By this definition: **d can't equal any f(i).** So: **f is not onto!**

Approach 1: mimic **power set** proof from Wednesday

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$f(4) = r_4 = 0.\ b_{41}\ b_{42}\ b_{43}\ b_{44} \ldots$

We're going to find a number **d** in the set (which is the codomain of f) that is not in this list

$$d_f = 0.\ b_1\ b_2\ b_3\ b_4 \ldots$$

where $b_i = 1 - b_{ii}$. By this definition: **d can't equal any f(i).** So: **f is not onto!**

Connection: associate each set of positive integers with a real number whose binary expansion has 1 as coefficient of 2^{-k} iff k is in the set.

Approach 2: Nested closed interval property

To show $f : \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ is not onto. **Strategy**: Build a sequence of nested closed intervals that each avoid some $f(n)$. Then the real number that is in all of the intervals can't be $f(n)$ for any n. Hence, f is not onto.

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Example: Consider the function $f : \mathbb{N} \to \{r \in \mathbb{R} \mid 0 \leq r \wedge r \leq 1\}$ with $f(n) = \frac{1+\sin(n)}{2}$

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n	$f(n)$	Interval that avoids $f(n)$
0	0.5	
1	0.920735...	
2	0.954649...	
3	0.570560...	
4	0.121599...	
5	0.020538...	
6	0.360292...	
7	0.828493...	
$\vdots$		

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0	0.5	$[0, 0.3]$ or ~~$[0.3, 0.6]$~~ or $[0.6, 1]$
1	$0.920735\ldots$	
2	$0.954649\ldots$	
3	$0.570560\ldots$	
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2	0.954649...	$[0.6, 0.63]$ or $[0.63, 0.66]$ or $[\mathbf{0.66}, \mathbf{0.7}]$
3	0.570560...	
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⋮		

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$\vdots$		

There is a real number, $\mathbf{d_f}$, that is in the intersection of all these nested intervals.

Claim: f is not onto because it misses d_f

Consequences

Arbitrary rational numbers have finite representations

Arbitrary reals always need to be approximated

Rational numbers & real numbers share many properties

The set of rational numbers is countably infinite

The set of real numbers is uncountable

The set of irrationals is $\mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} | x \notin \mathbb{Q}\}$ and is ...

A. Finite
B. Countably infinite
C. Uncountable