Countable sets

A set A is **finite** means it is empty or it is the same size as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$.

A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

A set A is **countable** means it is either finite or countably infinite.

Extra example Prove or disprove: There is a set Y, $\neg(\ |Y| = |Y \times Y|\ )$

Extra example Prove or disprove: There is a set Y, $\neg(\ |Y| = |\mathcal{P}(Y)|\ )$

$\mathbb{N}$ and its power set

Example elements of $\mathbb{N}$

Example elements of $\mathcal{P}(\mathbb{N})$

Recall: For set A, its power set is $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

Claim: $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$

Claim: There is an uncountable set. Example: ________

Proof: By definition of countable, since ________ is not finite, **to show** is $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$.

Rewriting using the definition of cardinality, **to show** is

Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

Rewriting using the definition of onto, **to show**:

$$\neg \forall B \in \mathcal{P}(\mathbb{N})\ \exists a \in \mathbb{N}\ (\ f(a) = B\ )$$

. By logical equivalence, we can write this as an existential statement:

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In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N}\ \mid\ n \notin f(n)\}$$

. By definition of power set, since all elements of D_f are in $\mathbb{N}$, $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N}\ (\ f(a) \neq D_f\ )$.
Proof of lemma:

By the Lemma, we have proved that f is not onto, and since f was arbitrary, there are no onto functions from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. QED

Where does D_f come from? The idea is to build a set that would "disagree" with each of the images of f about some element.

$n \in \mathbb{N}$	$f(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in D_f$?
0	$f(0) = X_0$	**Y / N**	Y / N	Y / N	Y / N	Y / N	...	**N / Y**
1	$f(1) = X_1$	Y / N	**Y / N**	Y / N	Y / N	Y / N	...	**N / Y**
2	$f(2) = X_2$	Y / N	Y / N	**Y / N**	Y / N	Y / N	...	**N / Y**
3	$f(3) = X_3$	Y / N	Y / N	Y / N	**Y / N**	Y / N	...	**N / Y**
4	$f(4) = X_4$	Y / N	Y / N	Y / N	Y / N	**Y / N**	...	**N / Y**
$\vdots$								