CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Today's learning goals

- Compare sizes of sets using one-to-one, onto, and invertible functions.
- Classify sets by cardinality into: **Finite sets, countable sets, uncountable sets.**
- Explain the central idea in Cantor's diagonalization argument.

|A| ≤ |B| means there is a one-to-one function from A to B.

$$\exists f : A \to B\ \forall a_1 \in A\ \forall a_2 \in A(\ a_1 \neq a_2 \ \to\ f(a_1) \neq f(a_2)\ )$$

|A| ≥ |B| means there is an onto function from A to B.

$$\exists f : A \to B\ \forall b \in B\ \exists a \in A(\ f(a) = b\ )$$

|A| = |B| means there is a bijection from A to B.

$$\exists f : A \to B\ \forall b \in B\ \exists a \in A(\ f(a) = b\ \wedge\ \forall a' \in A\ (a \neq a' \to f(a') \neq b)\ )$$

Cantor-Schroder-Bernstein Theorem:

|A| = |B| iff |A| ≤ |B| and |B| ≤ |A| iff |A| ≥ |B| and |B| ≥ |A|

Countable sets

Rosen Def 3 p. 171

Finite sets

|A| = |{1,...,n}| for some nonnegative int n

or A is the empty set

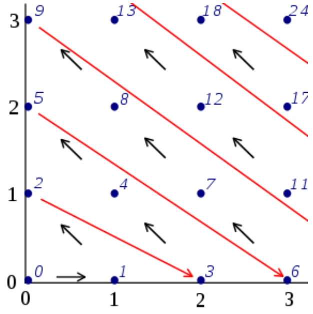
Countably infinite sets

"Smallest" infinite sets

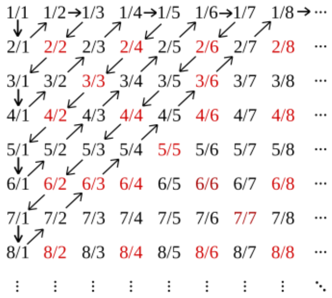
$|A| = |\mathbf{Z}^+|$ or $|A| = |\mathbf{N}|$ (e.g. can be listed out)

0 , -1 , 1 , -2 , 2 , -3 , 3 , …

$\mathbb{Z}$



$\mathbb{N} \times \mathbb{N}$



$\mathbb{Q}^+$

There is an uncountable set! *Rosen example 5, page 173-174*

"There are different sizes of infinity"

"Some infinities are smaller than other infinities"

Key insight: of all the set operations we've seen, the **power set** operation is the one where (for all finite examples) the output was a bigger set than the input.

The power set of a finite set is finite. What about the power set of an infinite set?

N and its power set

Which of the following are elements of $\mathcal{P}(\mathbb{N})$?

A. 0

B. $\emptyset$

C. {0}

D. $\{x \in \mathbb{N} \mid x \neq 0\}$

E. All of the above

N and its power set

Which of the following functions witness that $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$?

A. $f_A : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ where $f_A(x) = x^2$

B. $f_B : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ where $f_B(x) = \{x^2\}$

C. $f_C : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ where $f_C(x) = \{y \in \mathbb{N} \mid y \neq x^2\}$

D. All of the above

E. None of the above

N and its power set

Which of the following functions witness that $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{N}|$?

A. $g_A : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ where $g_A(x) = \begin{cases} 0 & \text{if } x = \emptyset \\ 1 & \text{otherwise} \end{cases}$

B. $g_B : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ where $g_B(x) = 20$

C. $g_C : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ where $g_C(x) = \begin{cases} y & \text{if } y \in x \\ 0 & \text{otherwise} \end{cases}$

D. All of the above

E. None of the above

N and its power set

Claim: $\mathcal{P}(\mathbb{N})$ is uncountable.

Towards a proof by universal generalization, consider an arbitrary function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

To show: f is not a bijection. It's enough to show that f is not onto.

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$$

. By definition of power set, since all elements of D_f are in $\mathbb{N}$, $D_f \in \mathcal{P}(\mathbb{N})$. It's enough to prove the following Lemma:

Lemma: $\forall a \in \mathbb{N}\ (\ f(a) \neq D_f\ )$.
Proof of lemma:

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$$

Where does this set come from?

Diagonalization

$$f_B : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \text{ where } f_B(x) = \{x^2\}$$

$n \in \mathbb{N}$	$f_B(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in D_{f_B}$?
0	$f_B(0) = \{0\} = X_0$	**Y**	N	N	N	N	...	No
1	$f_B(1) = \{1\} = X_1$	N	**Y**	N	N	N	...	No
2	$f_B(2) = \{4\} = X_2$	N	N	**N**	N	Y	...	Yes
3	$f_B(3) = \{9\} = X_3$	N	N	N	**N**	N	...	Yes
4	$f_B(4) = \{16\} = X_4$	N	N	N	N	**N**	...	Yes
$\vdots$								

In search of a witness, define the following collection of nonnegative integers:

$$D_f = \{n \in \mathbb{N} \mid n \notin f(n)\}$$

Where does this set come from?

Diagonalization

$$f_C : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \text{ where } f_C(x) = \{y \in \mathbb{N} \mid y \neq x^2\}$$

$n \in \mathbb{N}$	$f_C(n) = X_n$	Is $0 \in X_n$?	Is $1 \in X_n$?	Is $2 \in X_n$?	Is $3 \in X_n$?	Is $4 \in X_n$?	...	Is $n \in D_{f_C}$?
0	$f_C(0) = X_0$	**N**	Y	Y	Y	Y	...	Yes
1	$f_C(1) = X_1$	Y	**N**	Y	Y	Y	...	Yes
2	$f_C(2) = X_2$	Y	Y	**Y**	Y	N	...	No
3	$f_C(3) = X_3$	Y	Y	Y	**Y**	Y	...	No
4	$f_C(4) = X_4$	Y	Y	Y	Y	**Y**	...	No
$\vdots$								

Bonus Lemmas … how would you prove each one?

- If A and B are countable sets, then A U B is countable.

  Theorem 1, p. 174
- If A and B are countable sets, then A x B is countable.

  Generalize pairing idea
- If A is a subset of B, to show that |A| = |B|, it's enough to give a 1-1 function from B to A or an onto function from A to B.

  Exercise 22, p. 176
- If A is a subset of a countable set, then it's countable.

  Exercise 16, p. 176
- If A is a superset of an uncountable set, then it's uncountable.

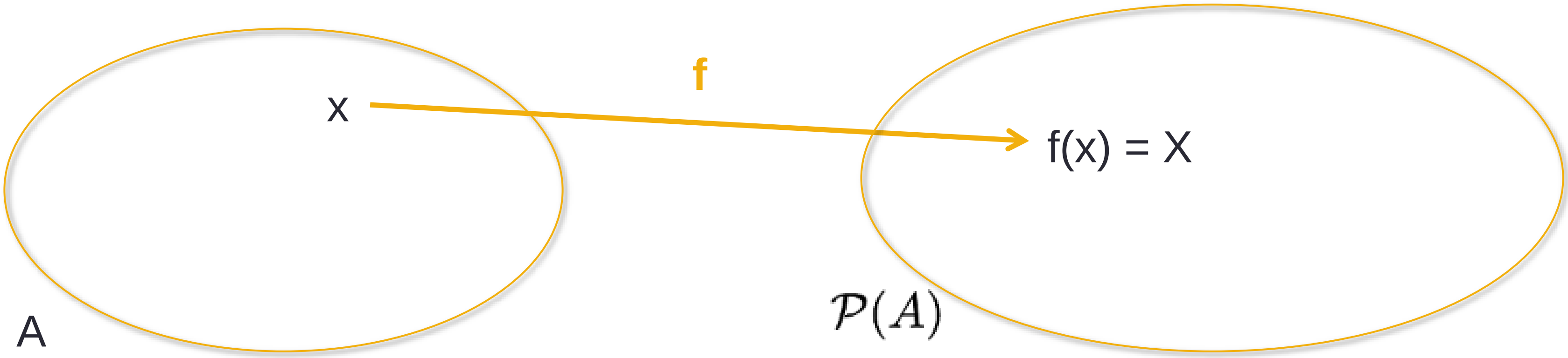
  Exercise 15, p. 176

There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Theorem: For every set A, $|A| \neq |\mathcal{P}(A)|$

Proof: (Proof by contradiction)



There is an uncountable set! *Rosen example 5, page 173-174*

Cantor's diagonalization argument

Consider the subset D of A defined by, for each a in A:

$$a \in D \qquad \text{iff} \qquad a \notin f(a)$$