For sets A and B

______________________ means $\exists f : A \to B\ \forall a_1 \in A\ \forall a_2 \in A(\ a_1 \neq a_2\ \to\ f(a_1) \neq f(a_2)\ )$

______________________ $\exists f : A \to B\ \forall b \in B\ \exists a \in A(\ f(a) = b\ )$

______________________ $\exists f : A \to B\ \forall b \in B\ \exists a \in A(\ f(a) = b\ \wedge\ \forall a' \in A\ (a \neq a' \to f(a') \neq b)\ )$

Cantor-Schroder-Bernstein Theorem: To prove $|A| = |B|$, we can do any **one** of the following

- Prove there exists a bijection $f : A \to B$;
- Prove there exists a bijection $f : B \to A$;
- Prove there exists two functions $f_1 : A \to B,\ f_2 : B \to A$ where each of f_1, f_2 is one-to-one.
- Prove there exists two functions $f_1 : A \to B,\ f_2 : B \to A$ where each of f_1, f_2 is onto.

Finite sets: A set A is **finite** means it is empty or it is the same size as $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$.

Countably infinite: A set A is **countably infinite** means it is the same size as $\mathbb{N}$.

Examples of countably infinite sets

Properties of cardinality

$$\forall A\ (\ |A| = |A|\ )$$
$$\forall A\ \forall B\ (\ |A| = |B|\ \to\ |B| = |A|\ )$$
$$\forall A\ \forall B\ \forall C\ (\ (|A| = |B|\ \wedge\ |B| = |C|)\ \to\ |A| = |C|\ )$$

Extra practice with proofs: Use the definitions of bijections to prove these properties.

More examples of countably infinite sets

Claim: S is countably infinite

One-to-one function from $\mathbb{Z}^+$ to S

One-to-one function from S to $\mathbb{N}$

Claim: L is countably infinite

One-to-one function from $\mathbb{N}$ to L

One-to-one function from L to $\mathbb{N}$

Claim: $|\mathbb{Z}^+| = |\mathbb{Q}|$

One-to-one function from $\mathbb{Z}^+$ to $\mathbb{Q}$

One-to-one function from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$

One-to-one function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}^+ \times \mathbb{Z}^+$

One-to-one function from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$