CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Today's learning goals

- Differentiate between important sets of numbers
- Compare sets using one-to-one, onto, and invertible functions.

|A| ≤ |B| means there is a one-to-one function from A to B.

$$\exists f : A \to B \; \forall a_1 \in A \; \forall a_2 \in A(\; a_1 \neq a_2 \; \to \; f(a_1) \neq f(a_2) \;)$$

|A| ≥ |B| means there is an onto function from A to B.

$$\exists f : A \to B \; \forall b \in B \; \exists a \in A(\; f(a) = b \;)$$

|A| = |B| means there is a bijection from A to B.

$$\exists f : A \to B \; \forall b \in B \; \exists a \in A(\; f(a) = b \; \wedge \; \forall a' \in A \; (a \neq a' \to f(a') \neq b) \;)$$

Cantor-Schroder-Bernstein Theorem:

|A| = |B| iff |A| ≤ |B| and |B| ≤ |A| iff |A| ≥ |B| and |B| ≥ |A|

Cardinality

Rosen Defn 3 p. 171

Finite sets — $|A| = |\{1,\ldots,n\}|$ for some nonnegative int n

Countably infinite sets — $|A| = |\mathbf{N}|$ (informally, can be listed out)

"Smallest" infinite set

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First 20 steps in enumeration.
Natural numbers
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
Positive integers
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Negative integers
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17 -18 -19 -20
Integers
0 -1 1 -2 2 -3 3 -4 4 -5 5 -6 6 -7 7 -8 8 -9 9 -10
```

Comparing cardinality

Countably infinite sets $|A| = |\mathbf{N}|$ (informally, can be listed out)

Which of the following sets is countably infinite?

A. $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$
B. $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$
C. $\mathbb{Z}^- = \{x \in \mathbb{Z} \mid x < 0\}$
D. $\mathbb{Z}$
E. All of the above

Live demo

CountablyInfinite.java

First 20 steps in enumeration.

Natural numbers
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
Positive integers
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Negative integers
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15 -16 -17 -18 -19 -20
Integers
0 -1 1 -2 2 -3 3 -4 4 -5 5 -6 6 -7 7 -8 8 -9 9 -10
Odd natural numbers
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39

CountablyInfinite.java

Consider the function $f : \_\_\_\_ \to \_\_\_\_$ given by $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$

This function can be used as witness that:

A. $|\mathbb{Z}^+| = |\mathbb{Z}^-|$

B. $|\mathbb{Z}^+| = |\mathbb{N}|$

C. $|\mathbb{Z}| = |\mathbb{N}|$

D. All of the above.

E. None of the above.

Properties of cardinality

$$\forall A \ (|A| = |A|)$$

$$\forall A \ \forall B(\ |A| = |B| \ \rightarrow \ |B| = |A| \ )$$

$$\forall A \ \forall B \ \forall C(\ (|A| = |B| \ \wedge \ |B| = |C|) \ \rightarrow \ |A| = |C| \ )$$

Corollary: All countably infinite sets are the same size as one another.

Comparing cardinality

Countably infinite sets $|A| = |\mathbf{N}|$ (informally, can be listed out)

Which of the following sets is countably infinite?

A. The set of ratings of 4 movies
B. The set of RNA strands
C. The set of hex colors
D. The set of linked lists of natural numbers
E. All of the above

Alphabetical* order

One-to-one function from $\mathbb{Z}^+$ to S

One-to-one function from S to $\mathbb{N}$

RNAStrand.java

Live demo

RNAStrand.java

Printing outputs of one-to-one function from set of positive integers to set of RNA strands

A, AA, AAA, AAAA, AAAAA, AAAAAA, AAAAAAA, AAAAAAAA, AAAAAAAAA, AAAAAAAAAA, AAAAAAAAAAA, AAAAAAAAAAAA, AAAAAAAAAAAAA, AAAAAAAAAAAAAA, AAAAAAAAAAAAAAA, AAAAAAAAAAAAAAAA, AAAAAAAAAAAAAAAAA, AAAAAAAAAAAAAAAAAA, AAAAAAAAAAAAAAAAAAA,

Printing RNA strands up to length 1 in alphabetical order
A, C, G, U,
Printing RNA strands up to length 2 in alphabetical order
A, C, G, U, AA, AC, AG, AU, CA, CC, CG, CU, GA, GC, GG, GU, UA, UC, UG, UU,

The natural number index of the RNA strand A is 0
The natural number index of the RNA strand CC is 9
The natural number index of the RNA strand UG is 18

RNAStrand.java

Another way – basis representation

- Alternative construction: one-to-one map from S to $\mathbb{N}$
- Strings: A,C,G,U,AA,AC,AG,AU,CA,CC,…
- View A,C,G,U as digits in base 10:
  - A=1, C=2, G=3, U=4

- Map: $A \to 1$, $AA \to 11$, $CA \to 21$, …

The set of linked lists

One-to-one function from $\mathbb{N}$ to L

One-to-one function from L to $\mathbb{N}$

Recall definition of L:
Basis Step: $[] \in L$
Recursive Step: If $l \in L$ and $n \in \mathbb{N}$, then $(n, l) \in L$

List.java

Live demo

List.java

Number for (0,[]): 1
Number for (1,[]): 2
Number for (2,[]): 4
Number for (3,[]): 8
Number for (0,(0,[])): 3
Number for (1,(3,[])): 13122
Number for (2,(1,[])): 36
Number for (2,(6,[])): 9223372036854775807
Number for (4,(2,[])): 1296
Number for (3,(9,[])): 9223372036854775807
Number for (6,(3,[])): 419904
Number for (0,(0,(0,[]))): 27
Number for (1,(2,(4,[]))): 9223372036854775807
Number for (3,(2,(1,[]))): 12007570823759930888

What happened here?

List.java

The set of rationals

$$\left\{ \frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

One-to-one function from $\mathbb{Z}^+$ to $\mathbb{Q}$

The set of rationals $\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$

One-to-one function from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$

One-to-one function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}^+ \times \mathbb{Z}^+$

One-to-one function from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$

Pairing

Live demo

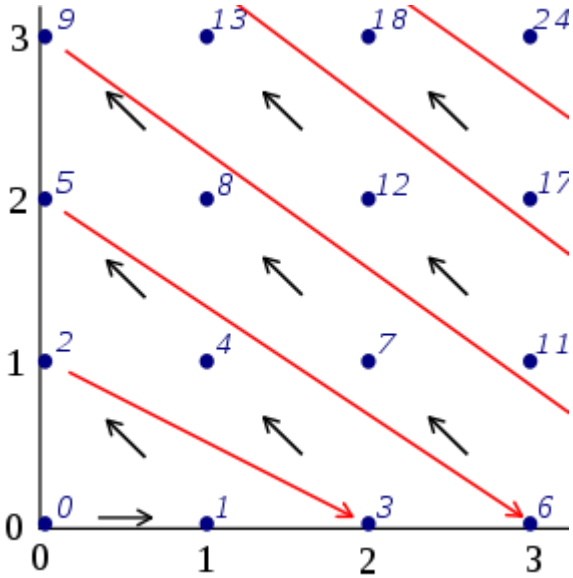
OrderedPair.java

Pairing

Number for (1, 1): 6
Number for (1, 2): 18
Number for (1, 3): 54
Number for (1, 4): 162
Number for (2, 1): 12
Number for (2, 2): 36
Number for (2, 3): 108
Number for (2, 4): 324
Number for (3, 1): 24
Number for (3, 2): 72
Number for (3, 3): 216
Number for (3, 4): 648

OrderedPair.java

Visually: countably infinite sets can be “listed out”



$\mathbb{N} \times \mathbb{N}$

1/1 1/2→1/3 1/4→1/5 1/6→1/7 1/8→ ⋯
2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 ⋯
3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 ⋯
4/1 4/2 4/3 4/4 4/5 4/6 4/7 4/8 ⋯
5/1 5/2 5/3 5/4 5/5 5/6 5/7 5/8 ⋯
6/1 6/2 6/3 6/4 6/5 6/6 6/7 6/8 ⋯
7/1 7/2 7/3 7/4 7/5 7/6 7/7 7/8 ⋯
8/1 8/2 8/3 8/4 8/5 8/6 8/7 8/8 ⋯
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋱

$\mathbb{Q}^+$

Recap

The **countably** infinite sets are the “smallest” infinite sets: they are in bijective correspondence with the set of natural numbers. They all have the same cardinality.

Many sets are countably infinite, including

- The set of all integers
- The set of positive integers
- The set of negative integers
- The set of odd integers
- The set of RNA strands
- The set of linked lists of natural numbers
- The set of rational numbers
- The set of ordered pairs of positive integers

For next time

Pre class reading: Example 5 Section 2.5 p173-174

** highly recommended **