CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Learning goals

Today's goals

- Evaluate which proof technique(s) is appropriate for a given proposition
- Compare sets using one-to-one, onto, and invertible functions.
- Define cardinality using one-to-one, onto, and invertible functions.

Sets of numbers

$\mathbb{Z}$	The set of integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^+$	The set of positive integers	$\{1, 2, \ldots\}$
$\mathbb{N}$	The set of nonnegative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	The set of rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} \text{ and } q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers	

$$\_\_\_ \subsetneq \_\_\_ \subsetneq \_\_\_ \subsetneq \_\_\_ \subsetneq \_\_\_$$

Subset inclusion is not the whole picture

Another approach: compare the **sizes of sets**

Finite sets

size of {1,2,3} is the same as **size** of {0,1,2} is the same as the **size** of {∅,π,√2}

size of {1,2,3} is less than or equal to the **size** of {A, U, C, G}

Infinite sets

How does the **size** of $\mathbb{N}$ compare to the **size** of $\mathbb{R}^+$?

How do we compare the sizes of (infinite) sets?

Key idea: functions let us associate elements of one set with another. If the association is "good" then we have a correspondence between (some) elements in one set with (some) elements of the other.

Use functions (with special properties) to relate the sizes of sets

functions

Rosen p. 139

Definition (Rosen p139): Let D and C be nonempty sets. A **function** f from D (domain) to C (codomain) is an assignment of one element of C to each element of D.

Which of these is an example of a well-defined function?

A. $f_A : \mathbb{R}^+ \to \mathbb{Q}$ $\quad f_A(x) = x$

B. $f_B : \mathbb{Q} \to \mathbb{Z}$ $\quad f_B\left(\frac{p}{q}\right) = p + q$

C. $f_C : \mathbb{Z} \to \mathbb{R}$ $\quad f_C(x) = \frac{x}{|x|}$ $\quad$ where $|\cdots|$ is absolute value

D. More than one of the above

E. None of the above

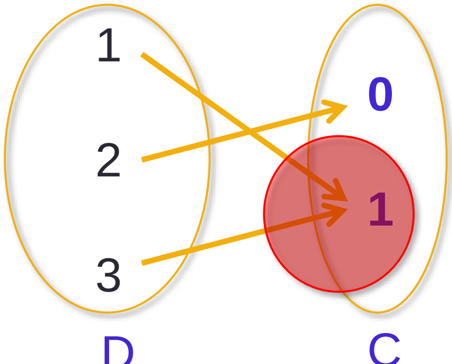
One-to-one **functions**

Rosen p. 141

Definition (Rosen p139): Let D and C be nonempty sets. A **function** f from D (domain) to C (codomain) is an assignment of one element of C to each element of D.

Definition (Rosen p141): A function $f : D \to C$ is **one-to-one** (or injective) means for every a, b in the domain, if $f(a) = f(b)$ then $a = b$.

A function f is **one-to-one** means **no duplicate images**



How can we formalize this?

A. $\forall a \in D\ \forall b \in D\ (f(a) \neq f(b))$

B. $\forall a \in D\ \forall b \in D\ (f(a) = f(b) \ \to\ a = b)$

C. $\forall a \in C\ \forall b \in C\ (a \neq b)$

D. $\forall a \in C\ \forall b \in C\ (a \neq b \ \to\ f(a) \neq f(b))$

E. None of the above

Cardinality

Definition: For sets A, B, we say that **the cardinality of A is no bigger than the cardinality of B**, and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B.

Analogy Seat assignments

Domain: Students in class
Codomain: Chairs in room

One-to-one function:

Well-defined function: each student is assigned one chair

- everyone has a seat
- no one is assigned two seats

Seat assignments can be made when there are no more students than chairs

Cardinality

Definition: For sets A, B, we say that **the cardinality of** A **is no bigger than the cardinality of** B, and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B.

Prove $|\{A, U, G, C\}| \leq |S_2|$, where S_2 is the set of RNA strands of length 2

CAUTION

Definition: For sets A, B, we say that **the cardinality of A is no bigger than the cardinality of B**, and write $|A| \leq |B|$, to mean there is a one-to-one function with domain A and codomain B.

This is the same symbol we use for comparing numbers but the definition and context are different!

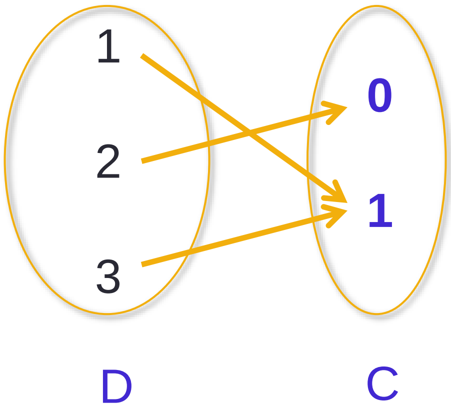
When A and B are finite sets, the definitions agree.

BUT, properties of numbers can't be assumed when A and B are infinite sets. *Stay tuned for next lectures…*

For now: what would >= mean?

Another way to compare size

Definition (Rosen p143): A function $f : D \to C$ is **onto** (or surjective) means for every b in the codomain, there is an element a in the domain with $f(a) = b$.



Analogy Exam versions

Domain: Students
Codomain: Versions

Onto function:

Well-defined function: each student is assigned one version
- everyone has an exam
- no one has two exams

No redundant versions when there are at least as many students as versions

Cardinality

Definition: For sets A, B, we say that **the cardinality of A is no smaller than the cardinality of B**, and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B.

Analogy Exam versions

Domain: Students
Codomain: Versions

Onto function:

Well-defined function: each student is assigned one exam version
- everyone has an exam
- no one has two exam versions

No redundant versions when there are at least as many students as versions

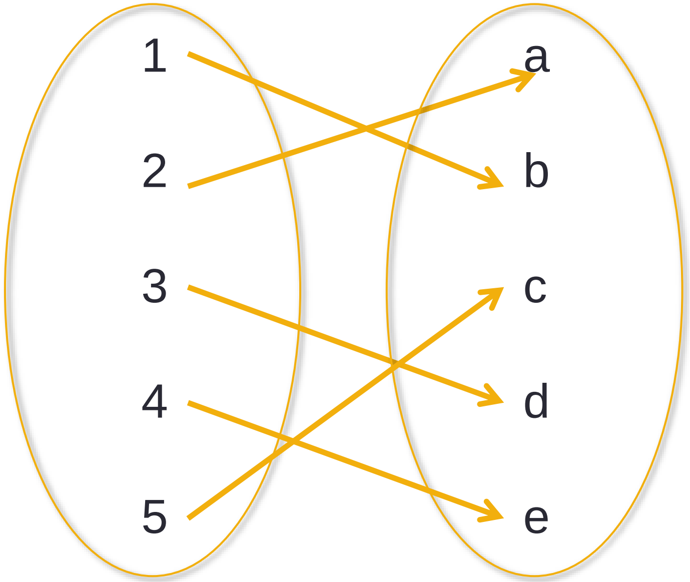
Cardinality

Definition: For sets A, B, we say that **the cardinality of** A **is no smaller than the cardinality of** B, and write $|A| \geq |B|$, to mean there is an onto function with domain A and codomain B.

Prove $|S_2| \geq |\{A, U, G, C\} \times \{A, U, G, C\}|$

One-to-one + onto

Rosen p. 144



one-to-one correspondence

bijection

invertible

Definition (Rosen p144): A function $f : D \to C$ is a **bijection** means that it is both one-to-one and onto. The **inverse** of a bijection $f : D \to C$ is the function $g : C \to D$ such that $g(b) = a$ iff $f(a) = b$.

Cardinality

Rosen Theorem 2, p 174

For nonempty sets A , B we say

|A| = |B| means there is a bijection from A to B.

Analogy Seat assignments

Domain: Students in class
Codomain: Chairs in room

One-to-one and onto function:

Well-defined function: each student is assigned one chair

- everyone has a seat
- no one is assigned two seats

Cardinality

Rosen Theorem 2, p 174

For nonempty sets A, B we say

$|A| \leq |B|$ means there is a one-to-one function from A to B.

$|A| \geq |B|$ means there is an onto function from A to B.

$|A| = |B|$ means there is a bijection from A to B.

These definitions all amount to comparing nonnegative integers when X is finite

Cantor-Schroder-Bernstein Theorem:

$$|A| = |B| \quad \text{iff} \quad |A| \leq |B| \text{ and } |B| \leq |A| \quad \text{iff} \quad |A| \geq |B| \text{ and } |B| \geq |A|$$

Recap

Functions can be defined by formula, table of values, recursively (depending on domain and codomain).

Use logical structure of definitions of one-to-one, onto to determine appropriate proof strategies.

Cardinality is defined via functions. This definition agrees with “size” when the sets are finite.

For next time

Pre class reading: Definition 3, Example 1 Section 2.5 p171

** highly recommended **