CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Learning goals

Today's goals

- Distinguish between and use as appropriate each of structural induction, mathematical induction, and strong induction

Mathematical induction

To prove a universal quantification where the element comes from the set of integers >= b, prove two cases:

1. Prove the property is true about the number b

2. Consider an arbitrary integer n greater than or equal to b, assume (as the **induction hypothesis**) that the property holds for n, and use this and other facts to prove that the property holds for n+1.

Structural induction

To prove a universal quantification where the element comes from a recursively defined set, prove two cases:

1. Assume the element is one of those from the basis step and prove the conclusion

2. Assume the element is one of those from the recursive step, and assume that the property holds for the elements used to build it, and prove the conclusion.

Change making

For which nonnegative integers n can we make change for n with coins of value 5 cents and 3 cents?

Small values:

Pattern:

Restating: We can make change for __________, we cannot make change for ______________, and

______________________________________*

A $\forall n \in \mathbb{Z}^{\geq 8}\ \exists x \in \mathbb{Z}^{\geq 8}\ \exists y \in \mathbb{Z}^{\geq 8}\ \ (5x + 3y = n)$

B $\forall n \in \mathbb{Z}^{\geq 8}\ \exists x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \ (5x + 3y = n)$

C $\exists n \in \mathbb{Z}^{\geq 8}\ \forall x \in \mathbb{Z}^{\geq 8}\ \forall y \in \mathbb{Z}^{\geq 8}\ \ (5x + 3y = n)$

D $\exists n \in \mathbb{Z}^{\geq 8}\ \forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \ (5x + 3y = n)$

$\mathbb{Z}^{\geq 8} = \{x \in \mathbb{Z} \mid x \geq 8\}$

Mathematical induction

To prove a universal quantification where the element comes from the set of integers >= b, prove two cases:

1. Prove the property is true about the number b

2. Consider an arbitrary integer n greater than or equal to b, assume (as the **induction hypothesis**) that the property holds for n, and use this and other facts to prove that the property holds for n+1.

Proof by mathematical induction, with $b = 8$

Basis step: WTS property is true about 8

Mathematical induction

To prove a universal quantification where the element comes from the set of integers >= b, prove two cases:

1. Prove the property is true about the number b

2. Consider an arbitrary integer n greater than or equal to b, assume (as the **induction hypothesis**) that the property holds for n, and use this and other facts to prove that the property holds for n+1.

Proof by mathematical induction, with $b = 8$

Basis step: WTS property is true about 8

Recursive step: Consider an arbitrary $n \geq 8$. Assume (as the IH) that there are nonnegative integers x, y such that $n = 5x + 3y$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$.

Key insight: can write "+1" in terms of 5 and 3. How?

Mathematical induction

To prove a universal quantification where the element comes from the set of integers >= b, prove two cases:

1. Prove the property is true about the number b

2. Consider an arbitrary integer n greater than or equal to b, assume (as the **induction hypothesis**) that the property holds for n, and use this and other facts to prove that the property holds for n+1.

Basis step: WTS property is true about 8

Recursive step: Consider an arbitrary $n \geq 8$. Assume (as the IH) that there are nonnegative integers x, y such that $n = 5x + 3y$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$. We consider two cases, depending on whether any 5 cent coins are used for n.

Case 1: Assume $x \geq 1$. Define $x' = x - 1$ and $y' = y + 2$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$5x' + 3y' \overset{\text{by def}}{=} 5(x-1) + 3(y+2) = 5x - 5 + 3y + 6$$
$$\overset{\text{rearranging}}{=} (5x + 3y) - 5 + 6$$
$$\overset{\text{IH}}{=} n - 5 + 6 = n + 1$$

Case 2: Assume $x = 0$. Therefore $n = 3y$, so since $n \geq 8$, $y \geq 3$. Define $x' = 2$ and $y' = y - 3$ (both in $\mathbb{N}$ by case assumption). Calculating:

$$5x' + 3y' \overset{\text{by def}}{=} 5(2) + 3(y-3) = 10 + 3y - 9$$
$$\overset{\text{rearranging}}{=} 3y + 10 - 9$$
$$\overset{\text{IH and case}}{=} n + 10 - 9 = n + 1$$

New approach: Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis** that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Key insight: once we have enough basis cases, we can represent larger numbers using ***one more*** *3-cent coin than a smaller number*

New approach: Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis** that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Proof by strong induction, with $b = 8$ and $j = 2$.

Basis step: WTS property is true about $8, 9, 10$

New approach: Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis** that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Recursive step: Consider an arbitrary $n \geq 10$. Assume (as the IH) that the property is true about each of $8, 9, 10, \ldots, n$. WTS that there are nonnegative integers x', y' such that $n + 1 = 5x' + 3y'$.

Recall: base expansion

Definition (Rosen p. 246) For b an integer greater than 1 and n a positive integer, the **base b expansion of n** is

$$(a_{k-1} \cdots a_1 a_0)_b$$

where k is a positive integer, $a_0, a_1, \ldots, a_{k-1}$ are nonnegative integers less than b, $a_{k-1} \neq 0$, and

$$n = a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

How do we prove that these coefficients **exist** for every positive integer n?

Recall: base expansion

Definition (Rosen p. 246) For n a positive integer, the **binary expansion** of n is $(a_{k-1} \cdots a_1 a_0)_2$ where k is a positive integer, $a_0, a_1, \ldots, a_{k-1}$ are each 0 or 1, $a_{k-1} \neq 0$, and $n = a_{k-1}2^{k-1} + \cdots + a_1 2 + a_0$.

How do we prove that these coefficients **exist** for every positive integer n?

Theorem: Every positive integer is a sum of (one or more) distinct powers of 2. *binary expansions exist!*

Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis**) that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Proof by strong induction, with $b = 1$ and $j = 0$.

Basis step: WTS property is true about 1.

Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis**) that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Recursive step: Consider an arbitrary integer $n \geq 1$.
Assume (as the IH) that the property is true about each of $1, \ldots, n$.
WTS that the property is true about $n + 1$.

A. By the IH, we can write n as a sum of distinct powers of 2. 1 is a power of 2. Add the term 1 to the sum for n to get n+1 as a sum of distinct powers of 2.

B. By the IH, we can write (n+1) **div** 2 as a sum of distinct powers of 2 (because (n+1) **div** 2 is positive and less than or equal to n). Increase the exponent in each term of this sum by 1. If (n+1) **mod** 2 is 1, add a new term to the sum: the term 2^0. The result sums to n+1.

C. Both of these work.
D. Neither of these work.

Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis**) that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Key insight: the base expansion of a number is a shifted version of the base expansion of half the number (with 0 or 1 at the end).

By the IH, we can write (n+1) **div** 2 as a sum of distinct powers of 2 (because (n+1) **div** 2 is positive and less than or equal to n). Increase the exponent in each term of this sum by 1. If (n+1) **mod** 2 is 1, add a new term to the sum: the term 2^0. The result sums to n+1.

A different way to represent positive integers

Definition (Rosen p257): An integer p greater than 1 is called **prime** if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called composite.

Theorem (Rosen p336): Every positive integer *greater than 1* is a product of (one or more) primes.

Before proving theorem, let's try some example prime factorizations. Which of these match the definitions?

A. 3 = 3
B. 100 = (1)(2)(2)(5)(5)
C. 20 = (4)(5)
D. 9 = (3)(3)
E. All of the above.

Strong induction

To prove a universal quantification where the element comes from the set of integers >= b:

1. Pick j basis cases and prove the property is true about b, …, b+j

2. Consider an arbitrary integer n that is >= b, assume (as the **strong induction hypothesis**) that the property holds for each of b, …, n, and use this and other facts to prove that the property holds for n+1.

Key insight: a number is either prime or is the product of two smaller numbers.

Theorem (Rosen p336): Every positive integer *greater than 1* is a product of primes.

Proof by strong induction, with $b = 2$ and $j = 0$.

Basis step: WTS property is true about 2.

Recursive step: Consider an arbitrary integer $n \geq 1$. Assume (as the IH) that the property is true about each of $1, \ldots, n$. WTS that the property is true about $n + 1$.
Case 1:

Case 2:

For next time

Reading for next time: Section 4.3 Example 2 Section (p. 258), Section 1.7 Example 9 (p. 86)