CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Learning goals

Today's goals

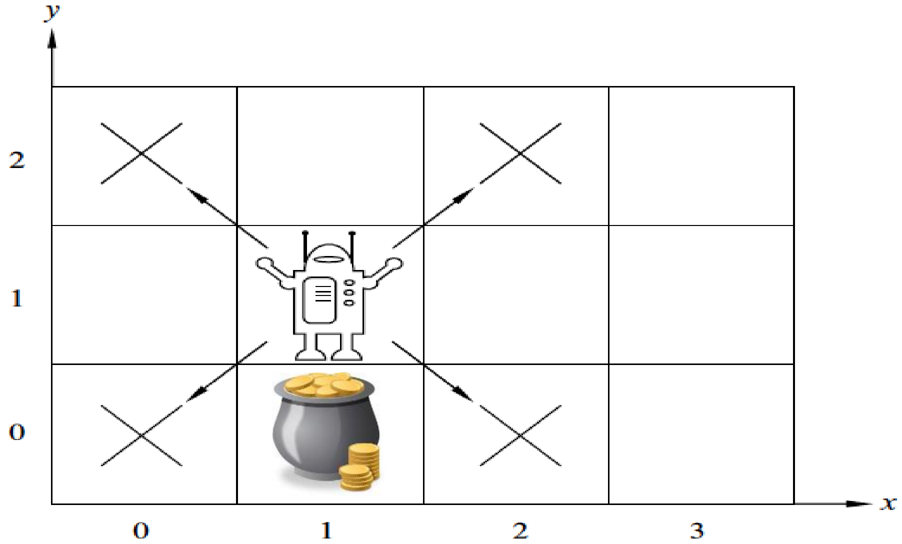
- Practice with properties of recursively defined sets and functions
- Prove and disprove properties of recursively defined sets and functions with structural induction and/or mathematical induction

Robot

Start at origin, moves on infinite 2-dimensional integer grid.
At each step, move to diagonally adjacent grid point.

Can it ever reach (1,0)?

A. Yes
B. No
C. I'd have to think about it more
D. I don't know



* Terminology: lemmas, theorems, corollaries, etc. on page 81 of textbook

Robot

Theorem:

Lemma*: The sum of the coordinates of any position reachable by the robot is even, i.e. has remainder 0 upon division by 2.

Using Lemma to prove the theorem…

* Terminology: lemmas, theorems, corollaries, etc. on page 81 of textbook

Robot

Theorem:

Lemma*: The sum of the coordinates of any position reachable by the robot is even, i.e. has remainder 0 upon division by 2.

Using Lemma to prove the theorem…

Let P be the set of coordinates the robot can reach. By Lemma, if $(x, y) \in P$ then $x + y$ is even. However, $1 + 0 = 1$ is odd so $(1,0) \notin P$.

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Robot

Lemma*: The sum of the coordinates of any position reachable by the robot is even, i.e. has remainder 0 upon division by 2.

Proof of Lemma: by structural induction on the set of possible positions of the robot.

Definition The set of positions the robot can visit P is defined by:

Basis Step:	$(0,0) \in P$
Recursive Step:	If $(x, y) \in P$, then $\boxed{}$ are also in P

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Robot

Lemma*: The sum of the coordinates of any position reachable by the robot is even, i.e. has remainder 0 upon division by 2.

Proof of Lemma: by structural induction on the set of possible positions of the robot.

Basis step: To show is 0+0 is an even integer.

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Robot

Lemma*: The sum of the coordinates of any position reachable by the robot is even, i.e. has remainder 0 upon division by 2.

Proof of Lemma: by structural induction on the set of possible positions of the robot.

Recursive step: Consider arbitrary (x,y) in P. To show is

(x+y is an even integer) → (sum of coordinates of next position is even integer)

Mathematical induction

To prove a universal quantification where the element comes from the set of positive integers, prove two cases:

1. Prove the property is true about the first number (usually 0 or 1)

2. Consider an arbitrary positive integer n, assume (as the **induction hypothesis** that the property holds for n, and use this and other facts to prove that the property holds for n+1.

Structural induction

To prove a universal quantification where the element comes from a recursively defined set, prove two cases:

1. Assume the element is one of those from the basis step and prove the conclusion

2. Assume the element is one of those from the recursive step, and assume that the property holds for the elements used to build it, and prove the conclusion.

Recall that the set of linked lists of natural numbers L Basis Step: $[] \in L$ Recursive Step: If $l \in L$ and $n \in \mathbb{N}$ then $(n, l) \in L$	Recall that length of a linked list of natural numbers L, $length : L \to \mathbb{N}$ is defined by: Basis step: $length([]) = 0$ Recursive step: If $l \in L$ and $n \in \mathbb{N}$ then $length((n, l)) = 1 + length(l)$

Prove or disprove: $\forall n \in \mathbb{N}\ \exists l \in L\ (\ length(l) = n\ )$

Growth of functions

Rosen p. 320

Definition The exponent function on $\mathbb{Z}^+$ is defined by:

$$\begin{array}{llll} \text{Basis Step:} & \text{If } n = 1 \text{ then} & 2^n & = 2 \\ \text{Recursive Step:} & \text{If } n \in \mathbb{Z}^+, \text{ then} & 2^{n+1} & = 2 \cdot 2^n \end{array}$$

Definition The factorial function on $\mathbb{Z}^+$ is defined by:

$$\begin{array}{llll} \text{Basis Step:} & \text{If } n = 1 \text{ then} & n! & = 1 \\ \text{Recursive Step:} & \text{If } n \in \mathbb{Z}^+, \text{ then} & (n+1)! & = (n+1) \cdot n! \end{array}$$

Which of the following is true?

A: $2^5 < 5!$
B: $3! < 2^3$
C: $2^2 < 2!$
D: $100! < 2^{100}$
E: More than one of the above

Prove or disprove: $\forall n \in \mathbb{Z}^+ \ (2^n < n!)$

Prove or disprove: $\forall n \in \mathbb{Z}^+ \ (2^n < n!)$

By mathematical induction:

Basis Step

Recursive Step. Consider . To show is:

$$\rightarrow$$

Assume **as the induction hypothesis, IH** that:

Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2$, and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

INDUCTIVE STEP: For the inductive step, we assume that $P(k)$ is true for an arbitrary integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$. We must show that under this hypothesis, $P(k+1)$ is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \geq 4$, then $2^{k+1} < (k+1)!$. We have

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\
&< 2 \cdot k! && \text{by the inductive hypothesis} \\
&< (k+1)k! && \text{because } 2 < k+1 \\
&= (k+1)! && \text{by definition of factorial function.}
\end{aligned}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

For next time

Pre class reading for next time: Example 2 Section 4.3 p258, Example 9 Section 1.7 p86