CSE 20
DISCRETE MATH

Fall 2020

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

Today's learning goals

- Use predicates with set of tuples as their domain to relate values to one another
- Evaluate nested quantifiers: both alternating and not.

Recall

A **predicate** is a function from a given set (domain) to {T,F}.

Cartesian product of sets A and B, A x B = { (a,b) | a in A and b in B}

Cartesian Products and Predicates

Definition (Rosen p123): Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}.$$

Recall: Each RNA strand is a string whose symbols are elements of the set $B = \{\texttt{A}, \texttt{C}, \texttt{G}, \texttt{U}\}$. The **set of all RNA strands** is called S. The function $rnalen$ that computes the length of RNA strands in S is:

		$rnalen : S$	$\rightarrow \mathbb{Z}^+$
Basis Step:	If $b \in B$ then	$rnalen(b)$	$= 1$
Recursive Step:	If $s \in S$ and $b \in B$, then	$rnalen(sb)$	$= 1 + rnalen(s)$

L with domain $S \times \mathbb{Z}^+$ is defined by, for $s \in S$ and $n \in \mathbb{Z}^+$,

$$L(s, n) = \begin{cases} T & \text{if } rnalen(s) = n \\ F & \text{otherwise} \end{cases}$$

Element where L evaluates to T: ________

Element where L evaluates to F: ________

Cartesian Products and Predicates

BC with domain $S \times B \times \mathbb{N}$ is defined by, for $s \in S$ and $b \in B$ and $n \in \mathbb{N}$,

$$BC(s,b,n) = \begin{cases} T & \text{if } basecount(s,b) = n \\ F & \text{otherwise} \end{cases}$$

Which of these is a witness that proves that

$$\exists t \; BC(t)$$

A. G
B. (GA, 2)
C. (GG, C, 0)
D. None of the above, but something else works.
E. None of the above, because the statement is false.

The **existential quantification** of $P(x)$ is the statement "There exists an element x in the domain such that $P(x)$" and is written $\exists x P(x)$. An element for which $P(x) = T$ is called a **witness** of $\exists x P(x)$.

Cartesian Products and Predicates

BC with domain $S \times B \times \mathbb{N}$ is defined by, for $s \in S$ and $b \in B$ and $n \in \mathbb{N}$,

$$BC(s,b,n) = \begin{cases} T & \text{if } basecount(s,b) = n \\ F & \text{otherwise} \end{cases}$$

Which of these is a counterexample that proves that

$$\forall (s,b,n) \; (\; BC(s,b,n) \;)$$

is false?

A. (G, A , 1)
B. (GC, A , 3)
C. (GG, G, 2)
D. None of the above, but something else works.
E. None of the above, because the statement is false.

The **universal quantification** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain" and is written $\forall x P(x)$. An element for which $P(x) = F$ is called a **counterexample** of $\forall x P(x)$.

Predicate	Domain	Example domain element where predicate is T
$basecount(s, b) = 3$		
$basecount(s, \texttt{A}) = n$		
$\exists n \in \mathbb{N}\ (basecount(s, b) = n)$		
$\forall b \in S\ (basecount(s, b) = 1)$		

Nested Quantifiers

$$\forall s \exists n \; BC(s, \mathtt{A}, n)$$

For **each** strand, there is some number such that the strand has that number of A's.

Nested Quantifiers

$\forall s \exists n \; BC(s, \mathtt{A}, n)$

For **each** strand, there is some number such that the strand has that number of A's.

$s = \mathtt{A}$	$s = \mathtt{U}$	$s = \mathtt{G}$	$\cdots$	$s = \mathtt{UGC}$	$\cdots$
$BC(\mathtt{A}, \mathtt{A}, n)$	$BC(\mathtt{U}, \mathtt{A}, n)$	$BC(\mathtt{G}, \mathtt{A}, n)$	$\cdots$	$BC(\mathtt{UGC}, \mathtt{A}, n)$	$\cdots$
$(\mathtt{A}, \mathtt{A}, 0)$	$(\mathtt{U}, \mathtt{A}, 0)$	$(\mathtt{G}, \mathtt{A}, 0)$		$(\mathtt{UGC}, \mathtt{A}, 0)$	
$(\mathtt{A}, \mathtt{A}, 1)$	$(\mathtt{U}, \mathtt{A}, 1)$	$(\mathtt{G}, \mathtt{A}, 1)$	$\cdots$	$(\mathtt{UGC}, \mathtt{A}, 1)$	$\cdots$
$(\mathtt{A}, \mathtt{A}, 2)$	$(\mathtt{U}, \mathtt{A}, 2)$	$(\mathtt{G}, \mathtt{A}, 2)$		$(\mathtt{UGC}, \mathtt{A}, 2)$	
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	

$$(\exists n \; BC(\mathtt{A}, \mathtt{A}, n)) \wedge (\exists n \; BC(\mathtt{C}, \mathtt{A}, n)) \wedge \exists n \; BC(\mathtt{G}, \mathtt{A}, n) \wedge \cdots \wedge (\exists n \; BC(\mathtt{UGC}, \mathtt{A}, n)) \wedge \cdots$$

Nested Quantifiers

$$\exists n \forall s \; BC(s, \mathtt{U}, n)$$

There is **some** number such that all strands have that number of U’s.

Nested Quantifiers

$$\exists n \forall s \; BC(s, \mathtt{U}, n)$$

There is some number such that all strands have that number of U's.

$n=0$		$n=1$		$n=2$		$\cdots$
$BC(s,\mathtt{U},0)$		$BC(s,\mathtt{U},1)$		$BC(s,\mathtt{U},2)$		$\cdots$
$(\mathtt{A},\mathtt{U},0)$	T	$(\mathtt{A},\mathtt{U},1)$	F	$(\mathtt{A},\mathtt{U},2)$	F	
$(\mathtt{U},\mathtt{U},0)$	F	$(\mathtt{U},\mathtt{U},1)$	T	$(\mathtt{U},\mathtt{U},2)$	F	
$(\mathtt{C},\mathtt{U},0)$	T	$(\mathtt{G},\mathtt{U},1)$	F	$(\mathtt{G},\mathtt{U},2)$	F	
$(\mathtt{G},\mathtt{U},0)$	T	$(\mathtt{C},\mathtt{U},1)$	F	$(\mathtt{C},\mathtt{U},2)$	F	
$(\mathtt{AA},\mathtt{U},0)$	T	$(\mathtt{AA},\mathtt{U},1)$	F	$(\mathtt{AA},\mathtt{U},2)$	F	
$(\mathtt{AU},\mathtt{U},0)$	F	$(\mathtt{AU},\mathtt{U},1)$	T	$(\mathtt{AU},\mathtt{U},2)$	F	
$(\mathtt{AG},\mathtt{U},0)$	T	$(\mathtt{AG},\mathtt{U},1)$	F	$(\mathtt{AG},\mathtt{U},2)$	F	
$\vdots$		$\vdots$		$\vdots$		
$(\mathtt{UCU},\mathtt{U},0)$	F	$(\mathtt{UCU},\mathtt{U},1)$	F	$(\mathtt{UCU},\mathtt{U},2)$	T	
$\vdots$		$\vdots$		$\vdots$		

$$(\forall s \; BC(s,\mathtt{U},0)) \vee (\forall s \; BC(s,\mathtt{U},1)) \vee (\forall s \; BC(s,\mathtt{U},2)) \vee \cdots$$

Evidence for nested quantifiers

More generally, for predicate P with domain $X \times Y$: The statement

$$\forall x \exists y P(x, y)$$

means every table for an assignment of x ($\forall x$) must have at least one witness row ($\exists y$). One subtable can serve as a counterexample to give evidence that this statement is false. On the other hand, the statement

$$\exists y \forall x P(x, y)$$

means some table for an assignment of y ($\exists y$) must have every row T ($\forall x$). One subtable can serve as a witness to give evidence that this statement is true.

What about $\forall x \forall y P(x, y)$, $\forall y \forall x P(x, y)$, $\exists x \exists y P(x, y)$, $\exists y \exists x P(x, y)$?

Evaluate each quantified statement as T or F.

$\forall s\ \forall b\ \exists n\ BC(s,b,n)$	$\forall s\ \forall n\ \exists b\ BC(s,b,n)$	$\forall b\ \forall n\ \exists s\ BC(s,b,n)$
$\exists s\ \forall b\ \exists n\ BC(s,b,n)$	$\forall s\ \exists n\ \forall b\ BC(s,b,n)$	$\exists b\ \exists n\ \forall s\ BC(s,b,n)$

Extra example: Write the negation of each of the statements above, and use De Morgan's law to find a logically equivalent version where the negation is applied only to the BC predicate (not next to a quantifier).

Nested Quantifiers

Challenge: write symbolically
"There are (at least) two different strands that have the same number of As"

Summary

- Cartesian products describe sets as combinations of elements from sets
- Predicates with sets of tuples as their domain can relate values to one another
- When quantifiers are *nested*, the order matters. We read left to right.
- When quantifiers are *nested*, we can visualize and interpret them in several ways
  - As nested tables, one for each value in the outermost quantification
  - As a conjunction or disjunction of other quantified statements

For next time

- Read website carefully

http://cseweb.ucsd.edu/classes/fa20/cse20-a/

- Next pre-class reading:
  - Section 1.5 Table 1 (p. 60)