Multiclass linear prediction

Multiclass classification

Of the classification methods we have studied so far, which seem inherently binary?

- Nearest neighbor?
- Generative models?
- Linear classifiers?

$$\hat{y} = \arg\max_{y \in \mathcal{Y}} p(x \mid y) p(y)$$

$$\hat{y} = \arg\max_{y \in \mathcal{Y}} w_y^\top x$$

From binary to multiclass logistic regression

Binary logistic regression: for $\mathcal{X} = \mathbb{R}^d$, classifier given by $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$:

$$\Pr(y = 1|x) = \frac{e^{w \cdot x + b}}{1 + e^{w \cdot x + b}}$$

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When the labels are $\mathcal{Y} = \{1, 2, \ldots, k\}$:
Specify a classifier by $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k \in \mathbb{R}$:

$$\Pr(y = j|x) = \frac{e^{w_j \cdot x + b_j}}{e^{w_1 \cdot x + b_1} + \ldots + e^{w_k \cdot x + b_k}}$$

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Prediction: given a point x, predict label

$$\arg\max_j \Pr(y = j|x) = \arg\max_j (w_j \cdot x + b_j)$$

Multiclass logistic regression

- **Label space**: $\mathcal{Y} = \{1, 2, \ldots, k\}$
- **Parametrized classifier**: $w_1, \ldots, w_k \in \mathbb{R}^d$, $b_1, \ldots, b_k \in \mathbb{R}$:

$$\Pr(y = j|x) = \frac{e^{w_j \cdot x + b_j}}{e^{w_1 \cdot x + b_1} + \ldots + e^{w_k \cdot x + b_k}}$$

- **Prediction**: given a point x, predict label $\arg\max_j (w_j \cdot x + b_j)$.
- **Learning**: Given: $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$.
  Find: $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k$ that maximize the likelihood

$$\prod_{i=1}^{n} \Pr(y^{(i)}|x^{(i)})$$

  Taking negative log gives a convex minimization problem.

Multiclass Perceptron

Setting: $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{1, 2, \ldots, k\}$

Model: $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k \in \mathbb{R}$

Prediction: On instance x, predict label $\arg\max_j (w_j \cdot x + b_j)$

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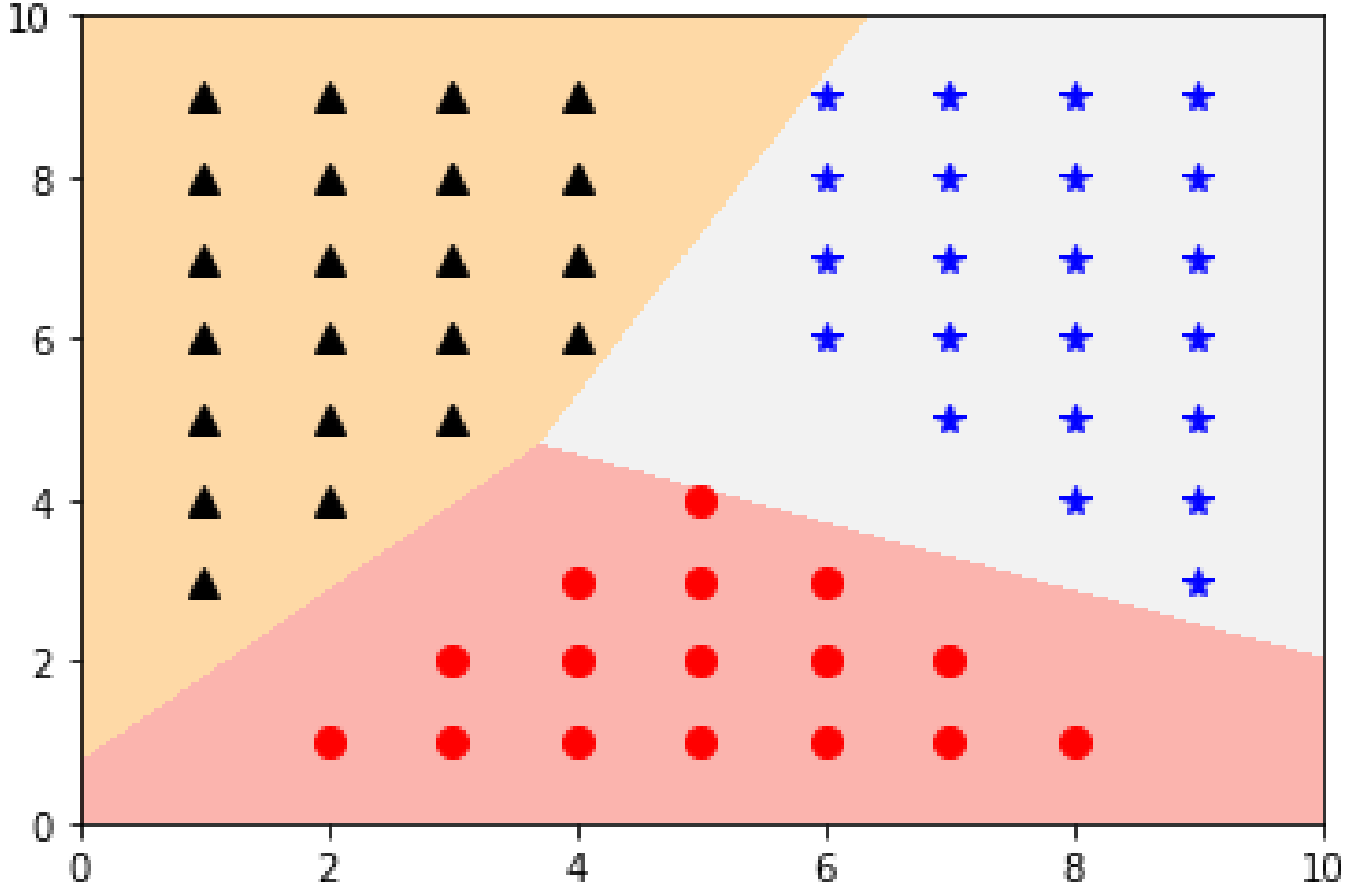
Learning. Given training set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$:

- Initialize $w_1 = \cdots = w_k = 0$ and $b_1 = \cdots = b_k = 0$
- Repeat while some training point (x, y) is misclassified:

$$\begin{aligned} \text{for correct label } y: \quad & w_y = w_y + x \\ & b_y = b_y + 1 \\ \text{for predicted label } \widehat{y}: \quad & w_{\widehat{y}} = w_{\widehat{y}} - x \\ & b_{\widehat{y}} = b_{\widehat{y}} - 1 \end{aligned}$$

Multiclass Perceptron: example

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Multiclass SVM

Model: $w_1, \ldots, w_k \in \mathbb{R}^d$ and $b_1, \ldots, b_k \in \mathbb{R}$

Prediction: On instance x, predict label $\arg\max_j (w_j \cdot x + b_j)$

Multiclass SVM

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Prediction: On instance x, predict label $\arg\max_j (w_j \cdot x + b_j)$

Learning. Given training set $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$:

$$\min_{w_1,\ldots,w_k \in \mathbb{R}^d, b_1,\ldots,b_k \in \mathbb{R}, \xi \in \mathbb{R}^n} \quad \frac{1}{2} \sum_{j=1}^{k} \|w_j\|^2 + C \sum_{i=1}^{n} \xi_i$$

$$w_{y^{(i)}} \cdot x^{(i)} + b_{y^{(i)}} - w_y \cdot x^{(i)} - b_y \geq 1 - \xi_i \quad \text{for all } i \text{ and all } y \neq y^{(i)}$$

$$\xi \geq 0$$

Once again, a convex optimization problem.

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Question: how many variables and constraints do we have?

Multiclass SVM example: iris

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